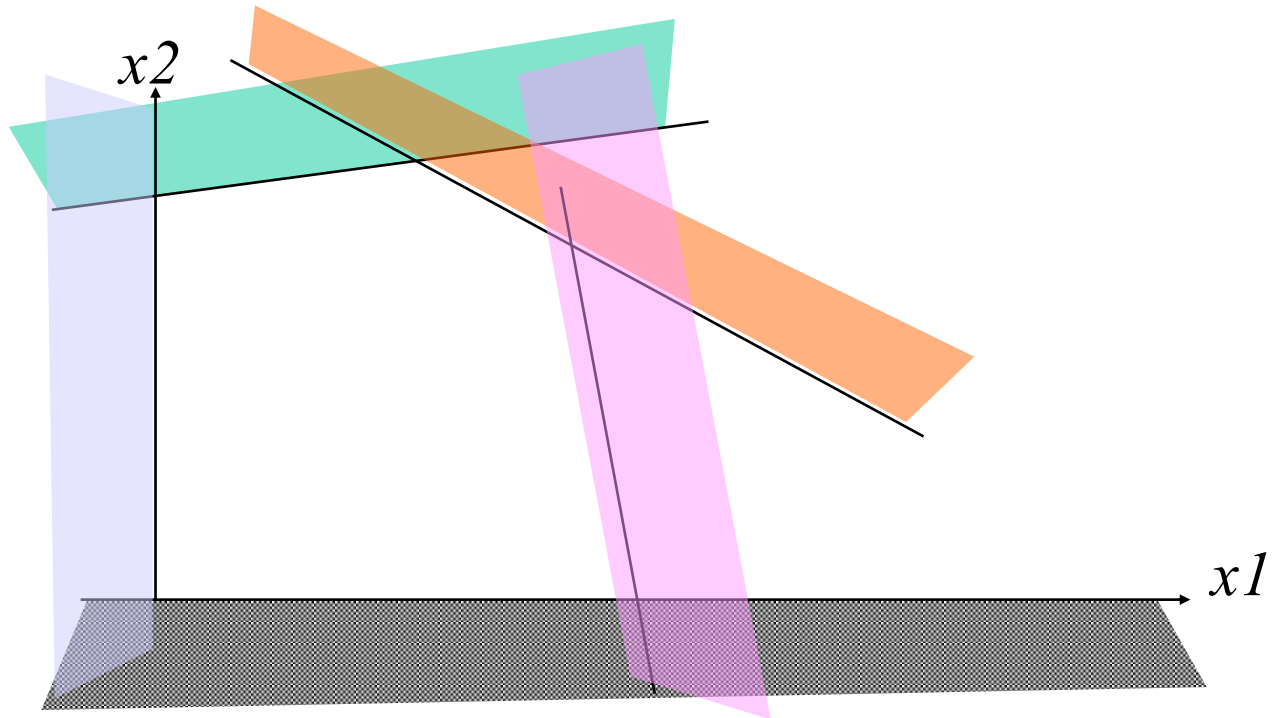


OPTIMIZATION SOLUTION METHODS

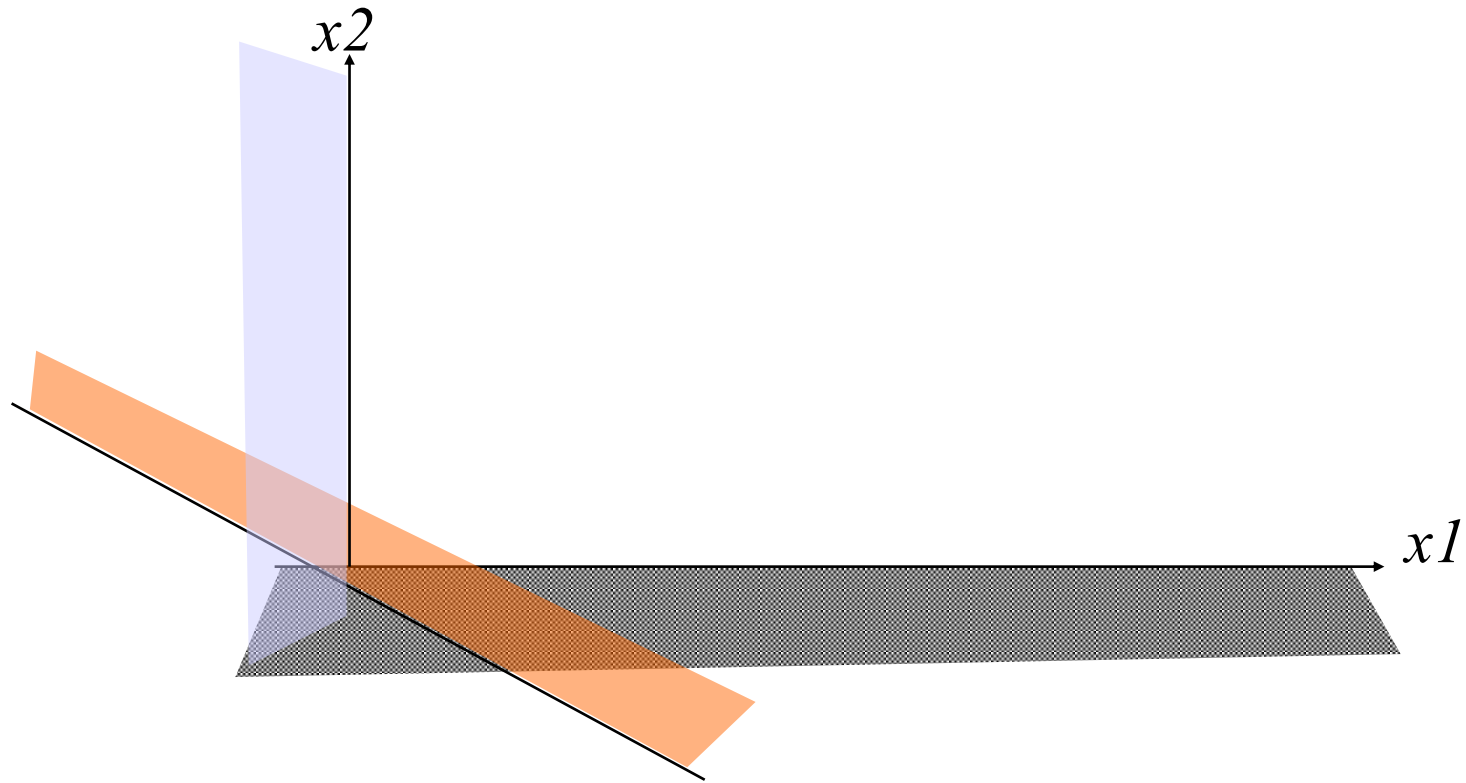
Solving linear programs

Well-formed Linear Programs



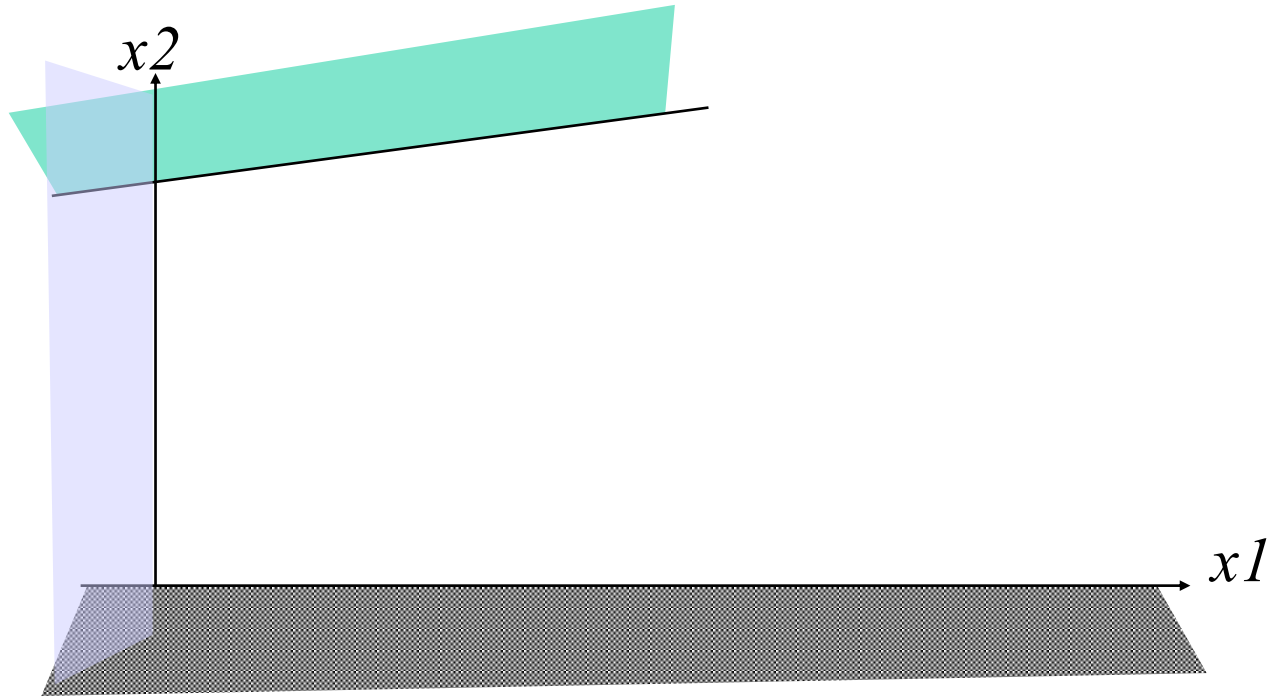
Feasible region is a **convex** polyhedron.

Ill-formed Linear Programs



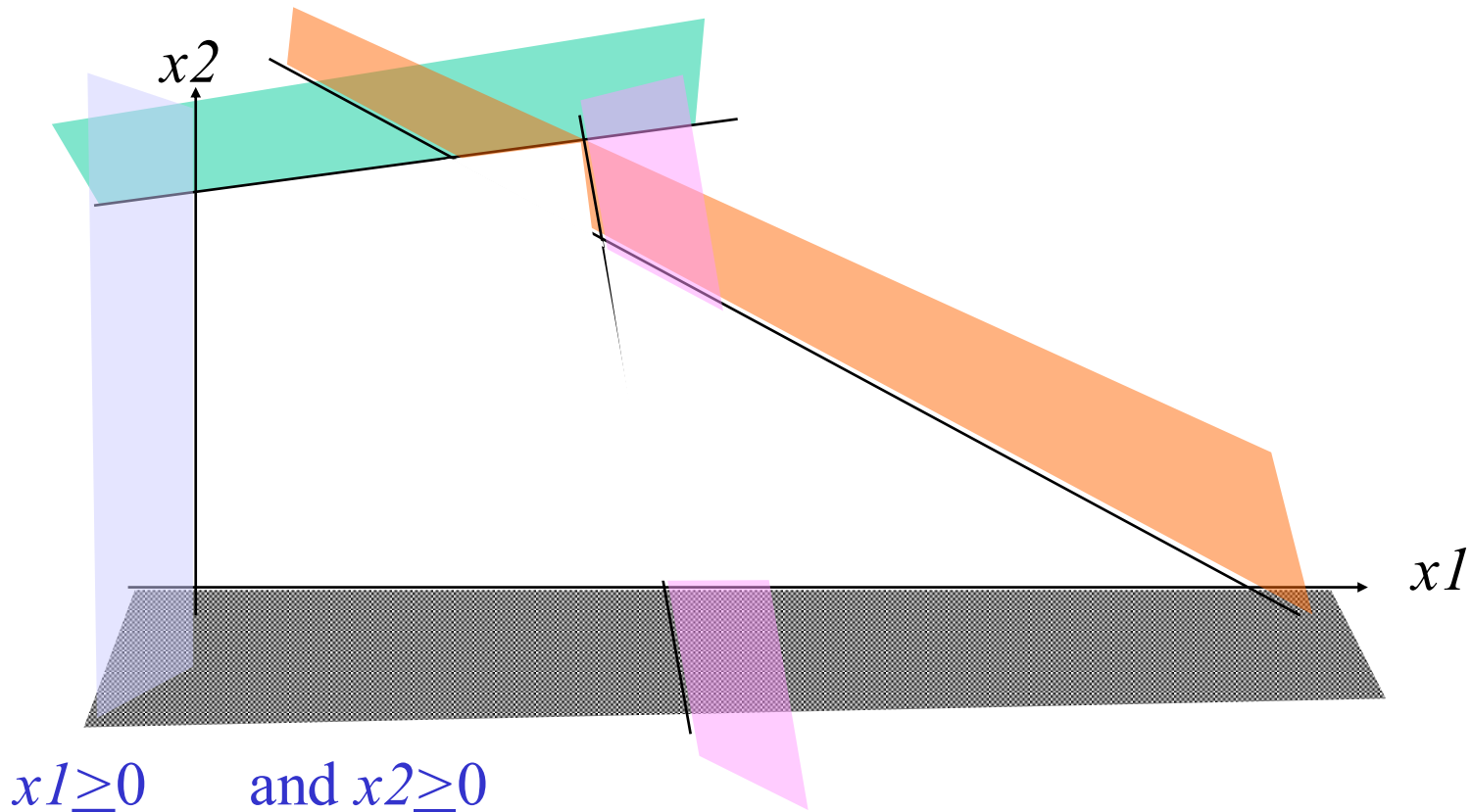
Empty feasible region (no solution).

Ill-formed Linear Programs



Unbounded feasible region (infinite utility).

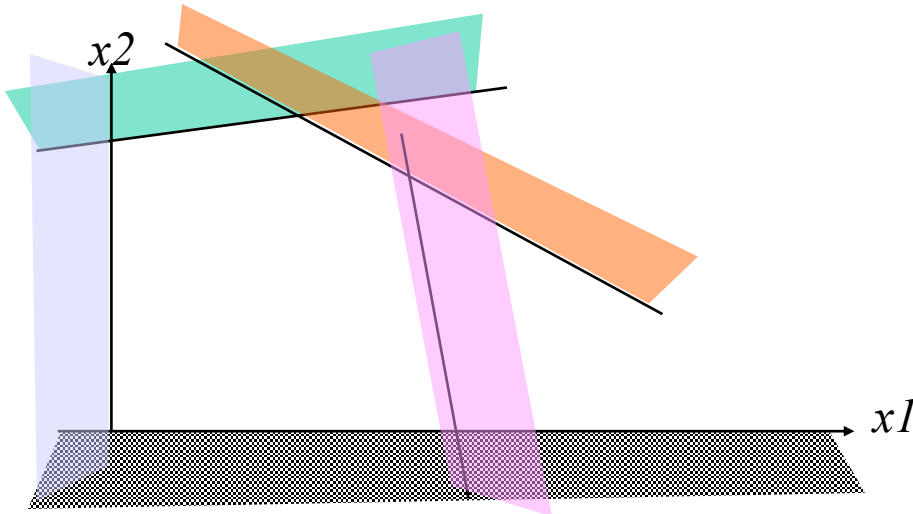
Ill-formed Linear Programs



Constraint 1 and
(Constraint 2
Or Constraint 3)

Non-convex feasible region.

Property: Optima at Corner Points



Standard Form:

$$\text{Maximize} \quad z = \mathbf{c}^T \mathbf{x}$$

$$\text{Subject to:} \quad \mathbf{Ax} \leq \mathbf{b}$$

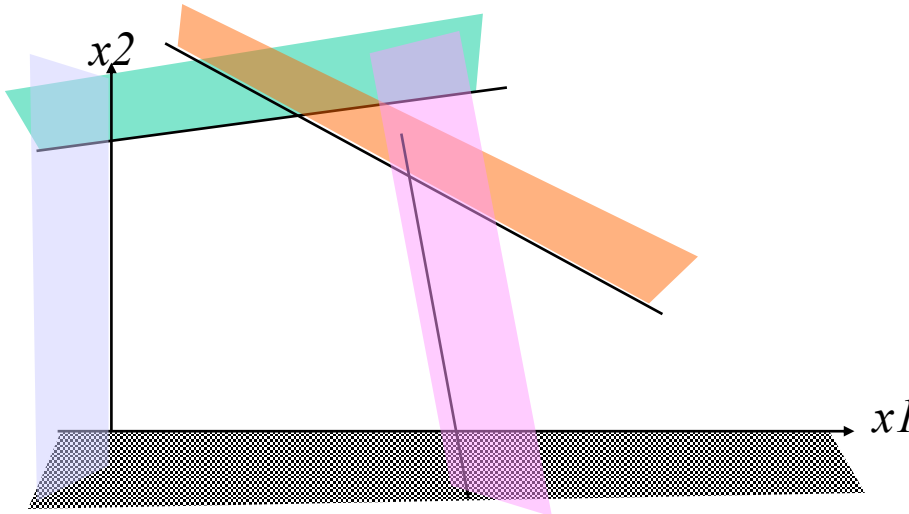
$$\mathbf{x} \geq \mathbf{0}$$

with n decision variables \mathbf{x}

An LP has no interior solutions! Why?

- An interior optima \mathbf{x}^* must satisfy $\square z(\mathbf{x}^*) = \mathbf{0}$
but for meaningful rewards $\square z(\mathbf{x}^*) = \mathbf{c} \square \mathbf{0}$

Property: Optima at Corner Points



Standard Form:

$$\text{Maximize } z = \mathbf{c}^T \mathbf{x}$$

$$\text{Subject to: } \mathbf{Ax} \leq \mathbf{b}$$

$$\mathbf{x} \geq 0$$

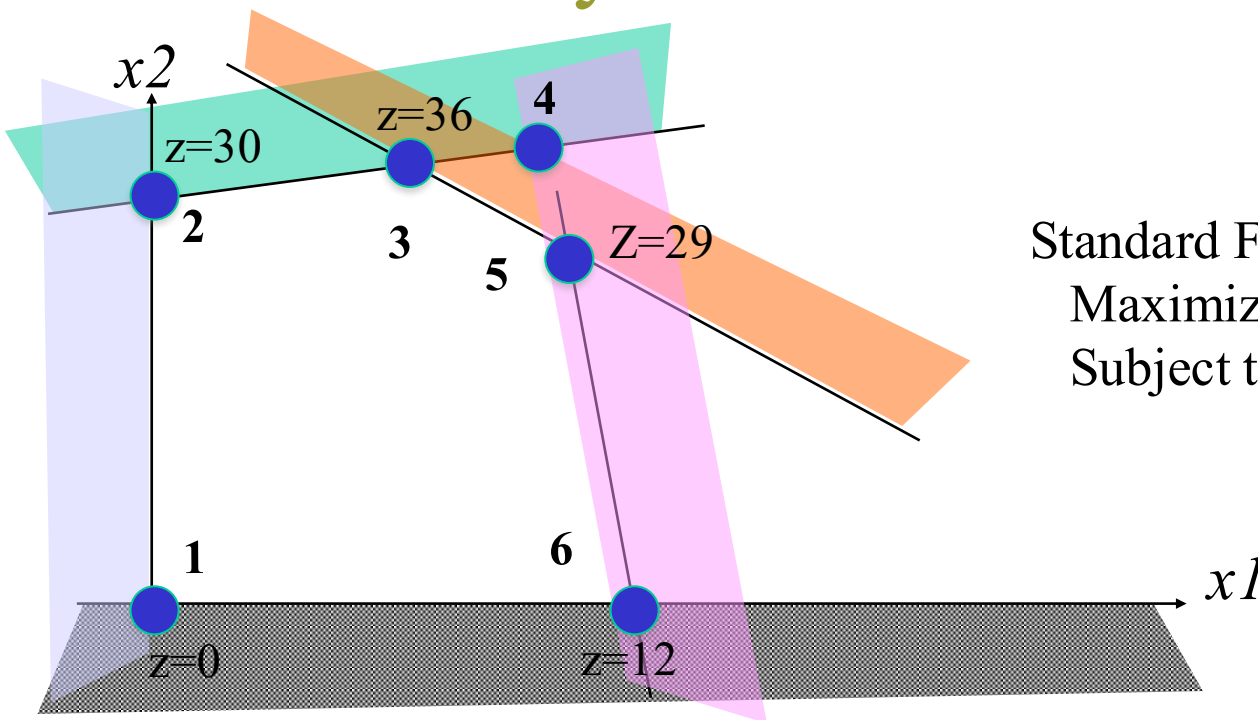
with n decision variables \mathbf{x}

Solutions are at corner points

- Point = intersection of n constraints
- Why?
 - Assume \mathbf{x}^* on constraint boundary B , but **not** at a **corner point**
 - Linear **objective increases** in **one direction** wrt B , unless constant
 - **Follow direction** until new **constraint** boundary **hit**
 - **Repeat** until **corner reached**, can't move further

Solve LPs by Generate and Test

corner points



Standard Form:

$$\begin{aligned} &\text{Maximize} && z = \mathbf{c}^T \mathbf{x} \\ &\text{Subject to:} && \mathbf{Ax} \leq \mathbf{b} \\ &&& \mathbf{x} \geq 0 \end{aligned}$$

Example:

$$\begin{aligned} \text{incumbent } \mathbf{x} &= \text{nil} && \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \text{incumbent } z &= -\text{inf} && 0 & 30 & 36 \end{aligned}$$

1. **incumbent** $\mathbf{x} = \text{nil}$, **incumbent** $z = -\text{inf}$
2. For each corner-point **cp**
 - If **cp** is feasible and z at **cp** $>$ **incumbent** z
Then **incumbent** $\mathbf{x} = \mathbf{cp}$, **incumbent** $z = z$ at **cp**
3. Return **incumbent** \mathbf{x} and z

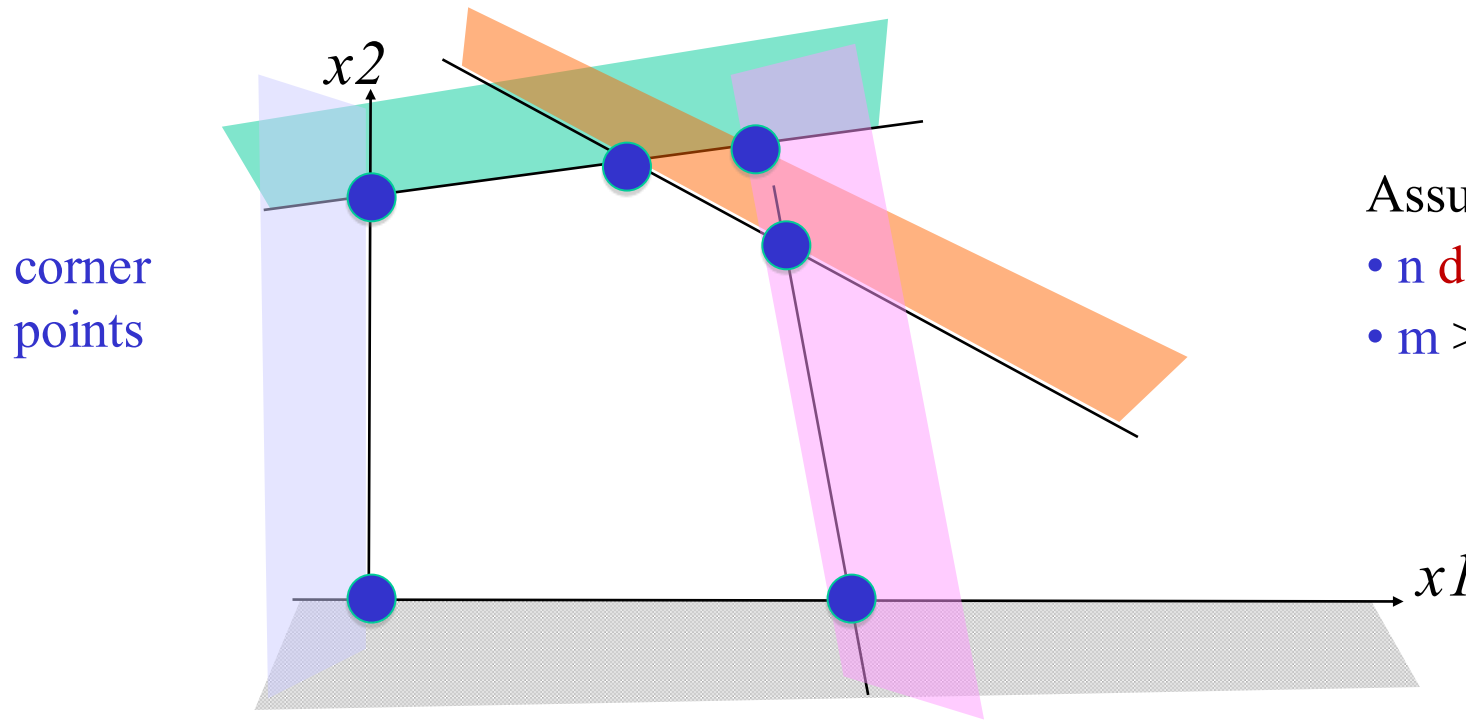


Solution

$$\begin{aligned} \mathbf{x} &= \mathbf{3} \\ z &= 36 \end{aligned}$$

9

Generating Corner Points

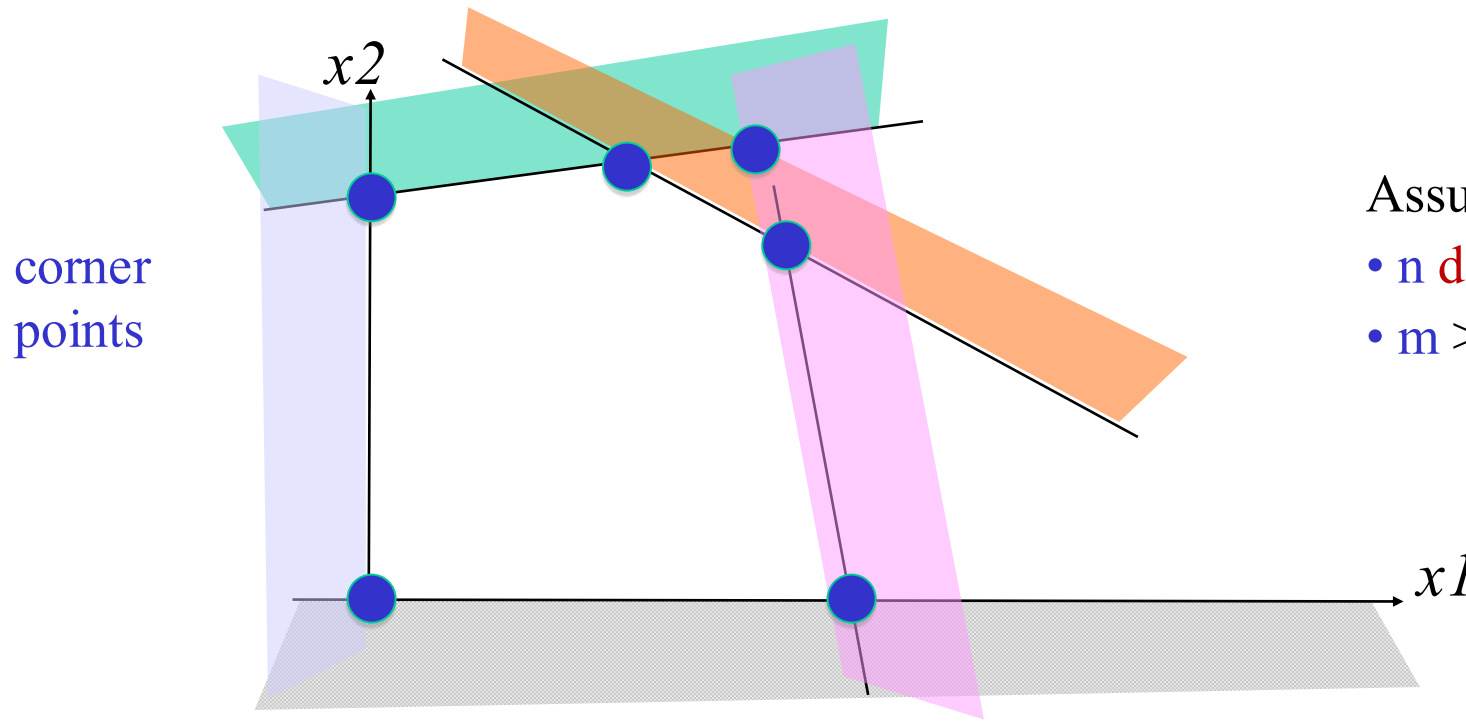


Assume:

- n decision variables
- $m > n$ inequalities

- Corner points are formed by activating n constraints
- To find:
 - Choose n active constraints using depth limited search with depth n
 - Find point by solving active constraints using Gaussian elimination $\Rightarrow O(n^3)$
 - Evaluate cost at each point

Generating Corner Points

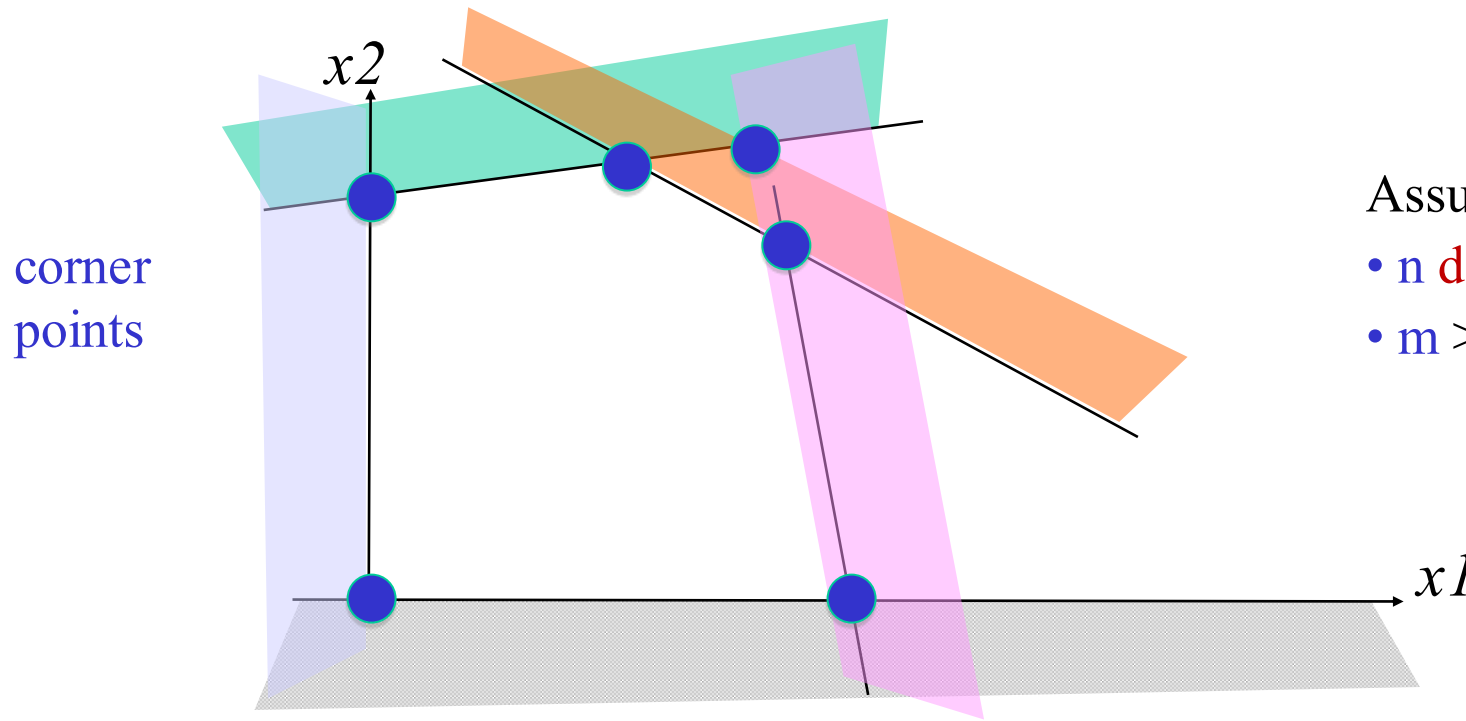


Assume:

- n decision variables
- $m > n$ inequalities

- Corner points are formed by activating n constraints $\Rightarrow O(n^3)$
- **Too many corner points!**
 - # corner points = $\binom{m}{n} = \frac{m!}{n!(m-n)!}$
 - 40 variables, 60 inequalities = 4,192 trillion points!

Generating Corner Points



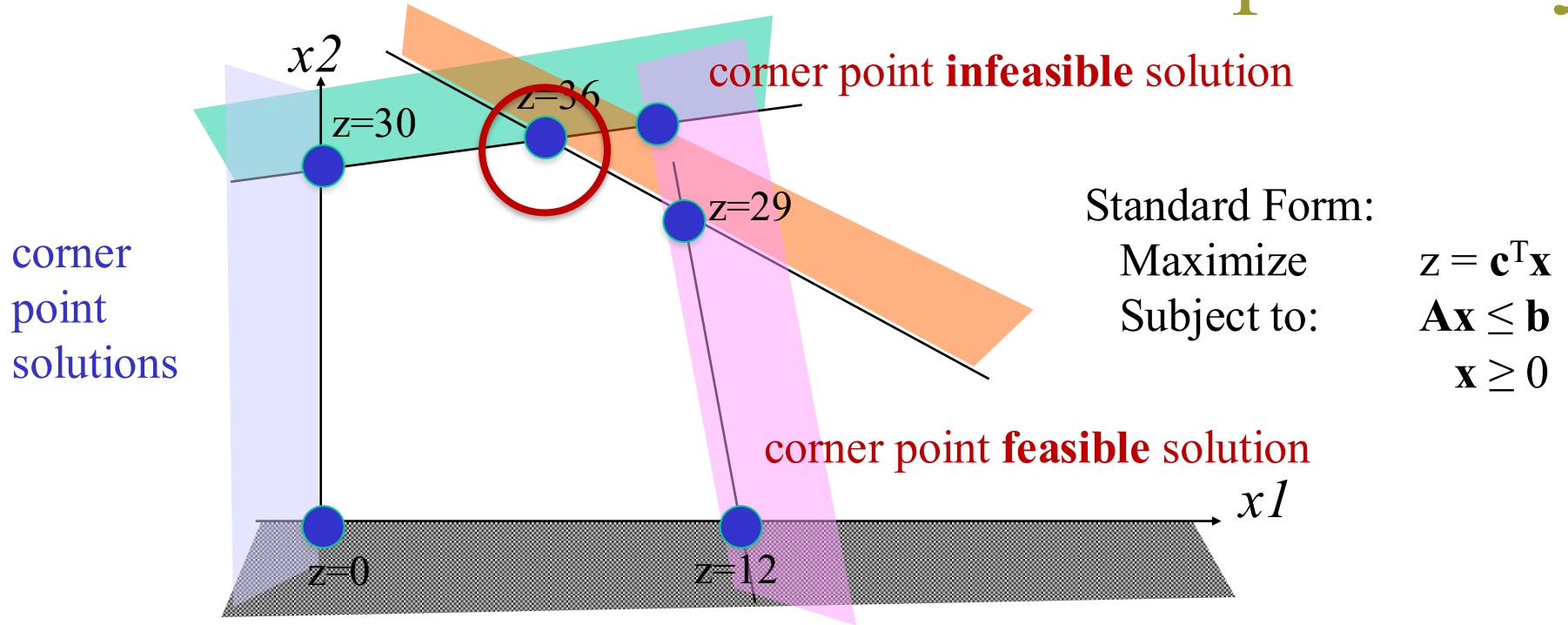
Assume:

- n decision variables
- $m > n$ inequalities

- Corner points are formed by activating n constraints $\Rightarrow O(n^3)$
- Too many \Rightarrow

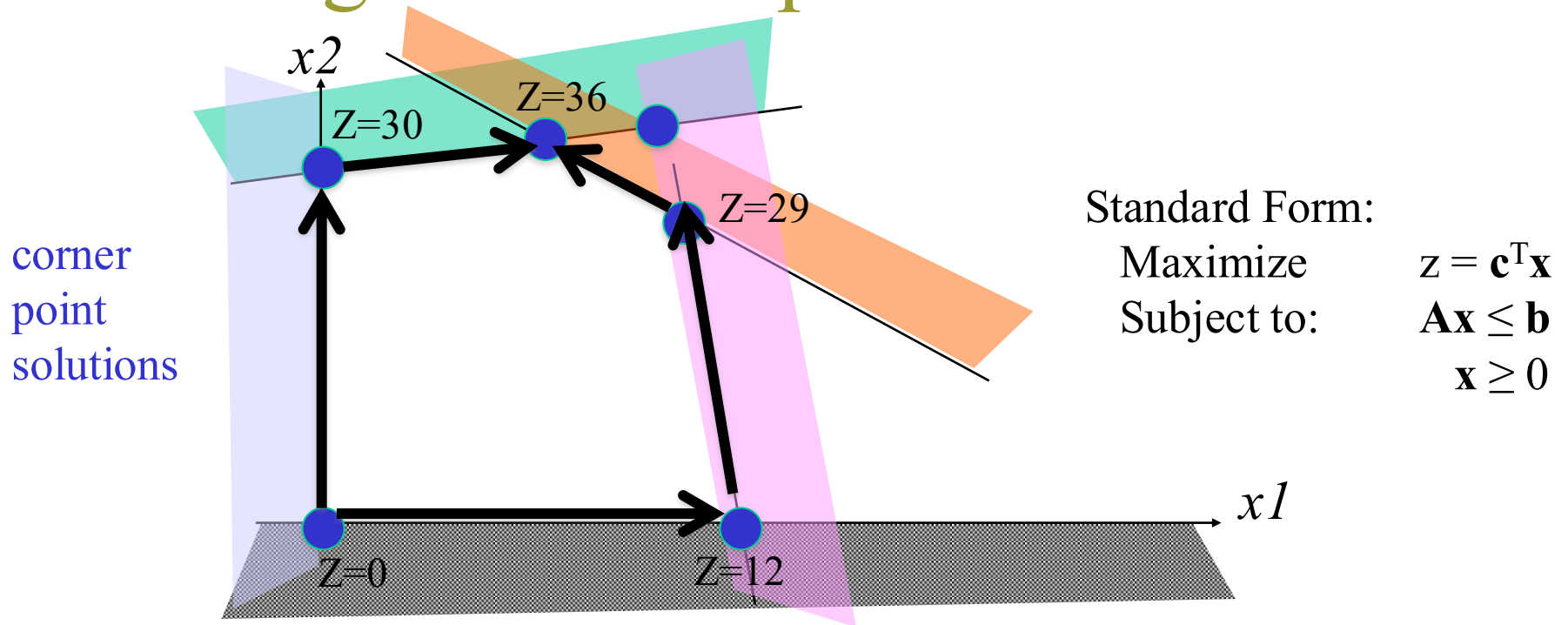
\square	m	\square	$=$	$\frac{m!}{n!(m-n)!}$
\square	n	\square		
- **How do we do better? (Simplex)**
 - Hill climb
 - Solve equations incrementally

Condition for Corner Point Optimality



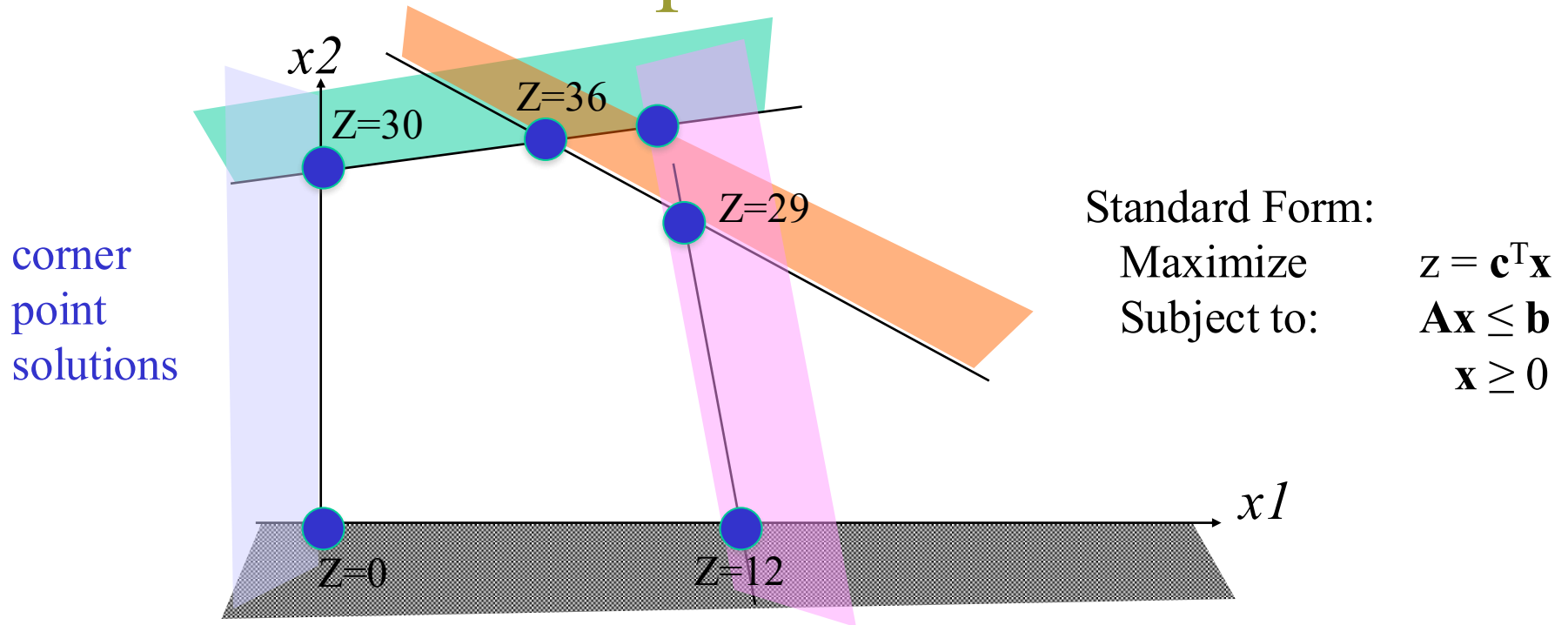
- If a **feasible corner-point** has **no better feasible neighbor**, it is **optimal**.
- If there are **multiple** optima, they are **neighbors**.

Finding Paths to Optimal Corner Points



- From every **feasible corner-point**, there is a **feasible path** to each **optima** *that never decreases cost*
- If we **move** from a **corner-point** to any **non-decreasing neighbor**, it has a **non-decreasing feasible path** to an **optima**

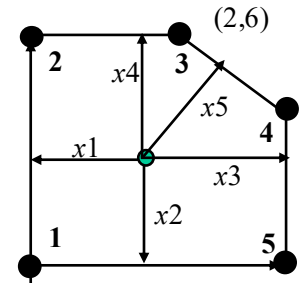
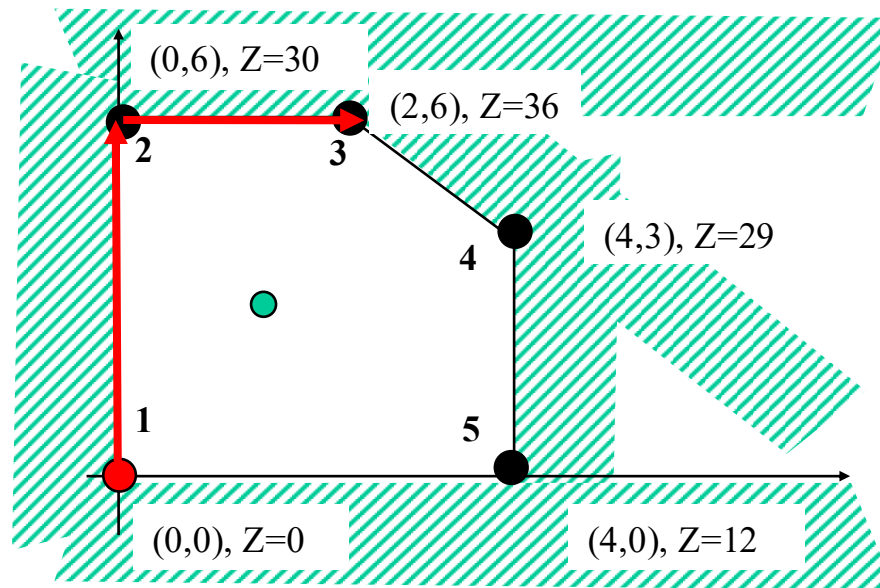
Simplex Method



1. *Current* = a feasible corner point (origin);
2. Repeat:
 - If *current* has **no** better neighbor,
 - Then **return** *current* as an **optima**,
 - Else new current = **neighbor** of *current* that **increases utility** (the most).

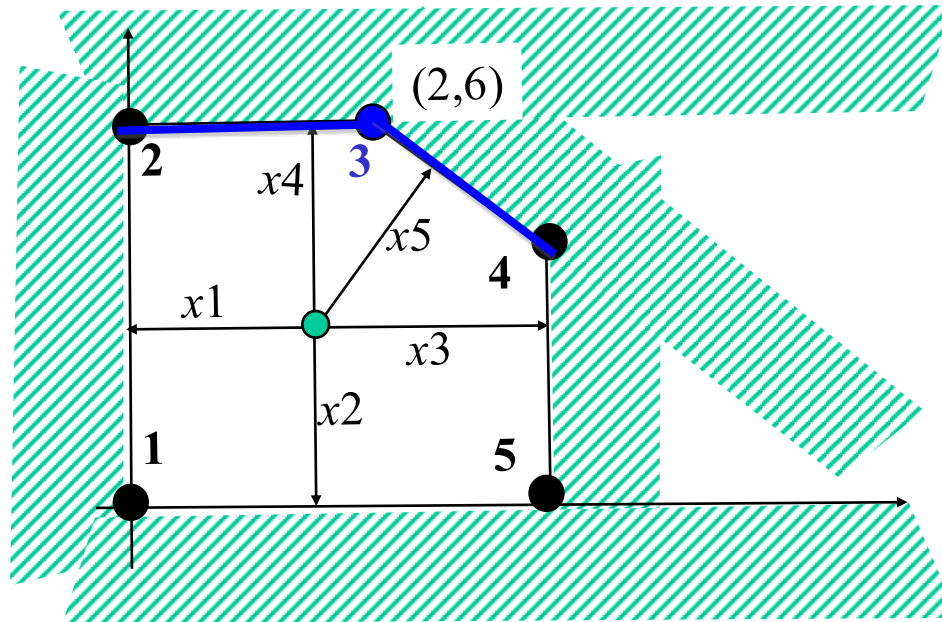
The Simplex Method

1. **Setup:** Add slack variables (augmented form)
2. **Initialize:** Start at a corner-point feasible solution
3. **Improve:** Move to a better neighboring feasible solution
4. **Optimality Test:** No better neighbor found



Finding Corner Points Incrementally

$$\begin{aligned} \text{Max } Z &= 3x_1 + 5x_2 \\ \text{Subject to: } &x_1 \leq 4 \\ &2x_2 \leq 12 \\ &3x_1 + 2x_2 \leq 18 \\ &x_1, x_2 \geq 0 \end{aligned}$$



Observation: Corner points **share most equations** \rightarrow **duplicate computation**

Goal: **Update** Gaussian elimination **incrementally**
after **activating** and **deactivating** constraints

Idea: To enable, manage **ALL constraints** in **one table**


\Rightarrow **Augmented Form**

To Find Corner Points Use Augmented Form

Idea: **Augmented Form (de)** activates constraints using “**slack**” variables \mathbf{x}'

➤ To **activate**, set “**slack**” to **zero**

$$\begin{aligned} \text{Max } Z &= 3x_1 + 5x_2 \\ \text{Subject to: } & x_1 \leq 4 \\ & 2x_2 \leq 12 \\ & 3x_1 + 2x_2 \leq 18 \\ & x_1, x_2 \geq 0 \end{aligned}$$

 $\mathbf{x}' = \langle x_3, x_4, x_5 \rangle$

$$\begin{aligned} \text{Max: } & Z \\ \text{Subject to: } & Z - 3x_1 - 5x_2 = 0 \\ & x_1 + x_3 = 4 \\ & 2x_2 + x_4 = 12 \\ & 3x_1 + 2x_2 + x_5 = 18 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

Standard Form:

$$\begin{aligned} \text{Max} & \quad z = \mathbf{c}^T \mathbf{x} \\ \text{Subject to:} & \quad \mathbf{Ax} \leq \mathbf{b} \\ & \quad \mathbf{x} \geq 0 \end{aligned}$$



Augmented Form:

$$\begin{aligned} \text{Max} & \quad z = \mathbf{c}^T \mathbf{x} \\ \text{Subject to:} & \quad z - \mathbf{c}^T \mathbf{x} = 0 \\ & \quad \mathbf{Ax} + \mathbf{x}' = \mathbf{b} \\ & \quad \underbrace{\mathbf{x}, \mathbf{x}'}_{\text{Augmented variables}} \geq 0 \end{aligned}$$

Augmented variables

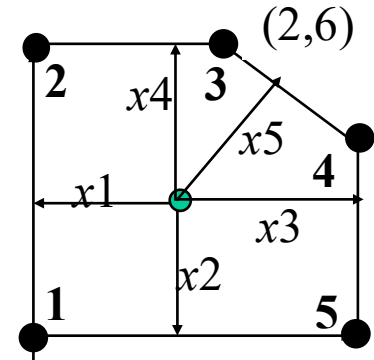
Activate Constraints of Corner Point by Zeroing their “Non-Basic” Variables

Non-Basic Variables: Set to zero → activates constraints

Basic Variables: → derived from constraints

To generate a corner point **CP** of dimension n :

1. Activate the n constraints of **CP**
 - Set their n “distance” variables to 0 (**non-basic**)
2. Solve for remaining variables (**basic**)



Point (1): At origin (0,0)

non-basic

$$\begin{array}{l} x1 = 0 \\ x2 = 0 \end{array}$$

basic

$$\begin{array}{l} x3 = 4 \\ x4 = 12 \\ x5 = 18 \end{array}$$

$$\begin{array}{l} x1 + x3 = 4 \\ 2x2 + x4 = 12 \\ 3x1 + 2x2 + x5 = 18 \end{array}$$

Point (3): At (2,6)

non-basic

$$\begin{array}{l} x4 = 0 \\ x5 = 0 \end{array}$$

basic

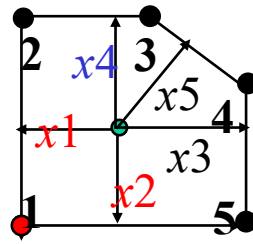
$$\begin{array}{l} x1 = 2 \\ x2 = 6 \\ x3 = 2 \end{array}$$

Uses same form for both points
How do we solve incrementally?

$$\begin{array}{l} x1 + x3 = 4 \\ 2x2 + x4 = 12 \\ 3x1 + 2x2 + x5 = 18 \end{array}$$

Initialize: Find Solution at Origin

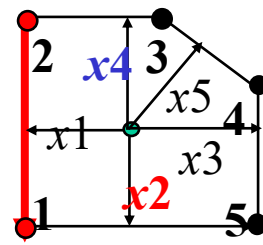
Non-basic: x_1, x_2 Basic: x_3, x_4, x_5



Note: Equations **already solved** for initial **basic**!

	z	x1	x2	x3	x4	x5	b
z	1	-3	-5				0
r1		1		1			4
r2			2		1		12
r3		3	2			1	18

First Iteration: Improve, . . .



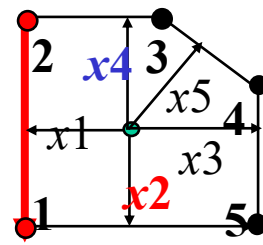
Non-basic: x_1, x_2 Basic: x_3, x_4, x_5

Idea: Move **along constraint** to **neighbor** that **improves most**, or stop if none.

	z	x1	x2	x3	x4	x5	b
z	1	-3	-5				0
r1		1		1			4
r2			2		1		12
r3		3	2			1	18

- **Pick non-basic** variable that **most improves Z**
- **Make basic**, by **increasing** until **blocked by constraint**, (shifts **constraint's variable** to **non-basic**)

First Iteration: Improve, . . .



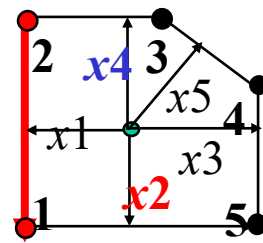
Non-basic: x_1, x_2 Basic: x_3, x_4, x_5

1. Choose column i (x_2)
with **most negative** entry
in **row z**

➤ Pick non-basic variable that **most improves Z**

	z	x1	x2	x3	x4	x5	b
z	1	-3	-5				0
r1		1		1			4
r2			2		1		12
r3		3	2			1	18

First Iteration: Improve, . . .

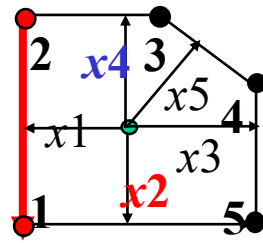


Non-basic: x_1 , ~~x_2~~ Basic: x_3 , x_4 , x_5

1. Choose column i (x_2)
with **most negative** entry
in **row z**

➤ Pick non-basic variable that **most improves Z**

	z	x_1	x_2	x_3	x_4	x_5	b
z	1	-3	-5				0
r_1		1		1			4
r_2			2		1		12
r_3		3	2			1	18



First Iteration: Improve, . . .

Non-basic: x_1 , x_2 Basic: x_3, x_4, x_5

1. Choose column i (x_2) with **most negative** entry in **row z**

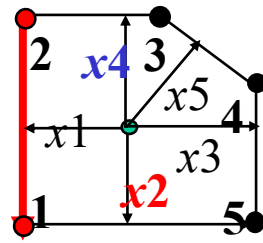
➤ Pick non-basic variable that **most improves Z**

➤ Make basic by **increasing** until **blocked** by constraint

2. Choose row j that **blocks** column i 's variable first (*Min Ratio Test* b_j / a_{ij})

	z	x_1	x_2	x_3	x_4	x_5	b
z	1	-3	-5				0
r_1		1	0	1			4
r_2			2		1		12
r_3		3	2			1	18

∞



First Iteration: Improve, . . .

Non-basic: x_1 , ~~x_2~~ Basic: x_3, x_4, x_5

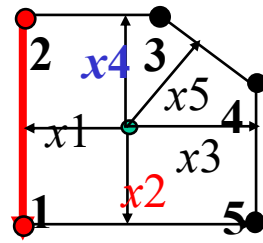
1. Choose column i (x_2) with **most negative** entry in **row z**

- Pick non-basic variable that **most improves Z**
- Make basic by **increasing** until **blocked** by constraint

2. Choose row j that **blocks** column i 's variable first (*Min Ratio Test* b_j / a_{ij})

	z	x1	x2	x3	x4	x5	b
z	1	-3	-5				0
r1		1		1			4
r2			2		1		12
r3		3	2			1	18

∞
6



First Iteration: Improve, . . .

Non-basic: x_1 , ~~x_2~~ Basic: x_3, x_4, x_5

1. Choose column i (x_2) with **most negative** entry in **row z**

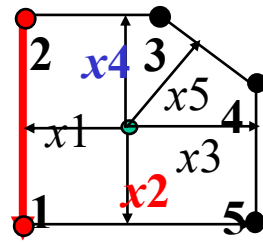
➤ Pick non-basic variable that **most improves Z**

➤ Make basic by **increasing** until **blocked** by constraint

2. Choose row j that **blocks** column i 's variable first (*Min Ratio Test* b_j / a_{ij})

	z	x_1	x_2	x_3	x_4	x_5	b	
z	1	-3	-5				0	
r_1		1		1			4	∞
r_2			2		1		12	6
r_3		3	2			1	18	9

⇒ pick r_2 , with slack x_4



First Iteration: Improve, . . .

Non-basic: x_1 , x_2 Basic: x_3 , x_4 , x_5

1. Choose column i (x_2) with **most negative** entry in **row z**

➤ Pick non-basic variable that **most improves Z**

➤ Make basic by **increasing** until **blocked** by constraint

2. Choose row j that **blocks** column i 's variable first

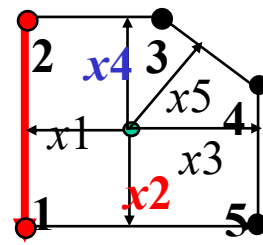
(*Min Ratio Test* b_j / a_{ij})

$x_2 \Rightarrow$ **basic**, $x_4 \Rightarrow$ **non-basic**

	z	x_1	x_2	x_3	x_4	x_5	b	
z	1	-3	-5				0	
r_1		1		1			4	∞
r_2			2		1		12	6
r_3		3	2			1	18	9

Non-basic: x_1 , x_4 Basic: x_2 , x_3 , x_5

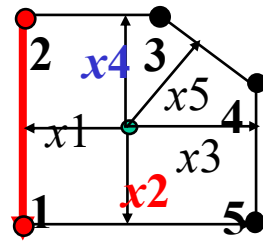
First Iteration: . . . Solve and Test



	z	x1	x2	x3	x4	x5	b
z	1	-3	0		5/2		30
r1		1		1			4
r2			1		1/2		6
r3		3	0		-1	1	6

4. **Optimality Test:**
 If entries of **z** **positive**,
no increase possible

Non-basic: x_1, x_4 **Basic:** x_2, x_3, x_5



Recap: First Iteration

Non-basic: x_1, x_2 Basic: x_3, x_4, x_5

1. Choose column i (x_2) with most negative entry in row z

- Pick non-basic variable that most improves Z
- Make basic by increasing until blocked by constraint

2. Choose row j (r_2) that first blocks an increase in the column's variable (Min Ratio Test b_j / a_{ij})

	z	x1	x2	x3	x4	x5	b
z	1	-3	-5				0
r1		1		1			4
r2			2		1		12
r3		3	2			1	18

$x_2 \Rightarrow$ basic, $x_4 \Rightarrow$ non-basic

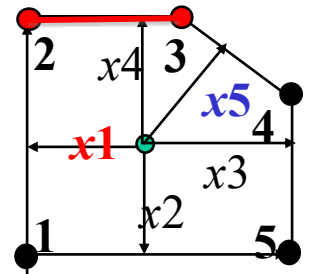
3. Pivot: Eliminate variable i (x_2), using row j (r_2), from other rows (z, r_1, r_3)
normalize

	z	x1	x2	x3	x4	x5	b
z	1	-3	0		5/2		30
r1		1		1			4
r2			1		1/2		6
r3		3	0		-1	1	6

4. Optimality Test: If entries of z positive, no increase possible

Non-basic: x_1, x_4 Basic: x_2, x_3, x_5

Second Iteration: Improve, Solve, Test



Non-basic: **x1, x4** Basic: **x2, x3, x5**

- Pick non-basic variable that **most improves Z**
- Make basic by **increasing** until **blocked** by constraint

Choose x1 as column with **most negative** entry in **row z**

	z	x1	x2	x3	x4	x5	b
z	1	-3	0		5/2		30
r1		1		1			4
r2			1		1/2		6
r3		3	0		-1	1	6

Choose row r3 as **first** to **block increase** in **x1**
(*Min Ratio Test* b_j/a_{ij})

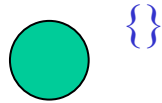
$x1 \Rightarrow$ **basic**, $x5 \Rightarrow$ **non-basic**
Pivot & normalize

	z	x1	x2	x3	x4	x5	b
z	1	0	0		3/2	1	36
r1		0		1	1/3	-1/3	2
r2			1		1/2		6
r3		1	0		-1/3	1/3	2

Non-basic: **x4, x5** Basic: **x1, x2, x3**

Solving linear programs with integer/discrete features

Example: B&B for BIPs



Solve:

$$\text{Max } Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

Subject to:

$$- 6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$$

$$- x_3 + x_4 \leq 1$$

$$- x_1 + x_3 \leq 0$$

$$- x_2 + x_4 \leq 0$$

$$- x_i \leq 1, x_i \geq 0, x_i \text{ integer}$$

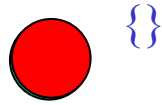
(DFS) Queue: {}

Incumbent: none

Best Z^* : - inf

• Initialize

Example: B&B for BIPs



Solve:

$$\text{Max } Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

Subject to:

$$- 6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$$

$$- x_3 + x_4 \leq 1$$

$$- x_1 + x_3 \leq 0$$

$$- x_2 + x_4 \leq 0$$

$$- x_i \leq 1, x_i \geq 0, x_i \text{ integer}$$

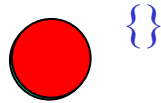
Queue: ~~{}~~

Incumbent: none

Best Z^* : - inf

• Dequeue $D_1 \equiv \{\}$

Example: B&B for BIPs



Solve:

$$\text{Max } Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

Subject to:

$$- 6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$$

$$- x_3 + x_4 \leq 1$$

$$- -x_1 + x_3 \leq 0$$

$$- -x_2 + x_4 \leq 0$$

$$- x_i \leq 1, x_i \geq 0, x_i \text{ ~~integer~~}$$

$$Z = \del{16.5}, x = \langle 0.8333, 1, 0, 1 \rangle$$

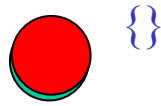
Queue:

Incumbent: none

Best Z^* : - inf

- Bound $D_1 \equiv \{$
 1. Constrain IP by $\}$
 2. Relax IP to LP
 3. Solve LP

Example: B&B for BIPs



Solve:

$$\text{Max } Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

Subject to:

$$- 6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$$

$$- x_3 + x_4 \leq 1$$

$$- -x_1 + x_3 \leq 0$$

$$- -x_2 + x_4 \leq 0$$

$$- x_i \leq 1, x_i \geq 0, x_i \text{ ~~integer~~}$$

$$Z = \mathbf{16.5}, \mathbf{x} = \langle \mathbf{0.8333}, \mathbf{1}, \mathbf{0}, \mathbf{1} \rangle$$

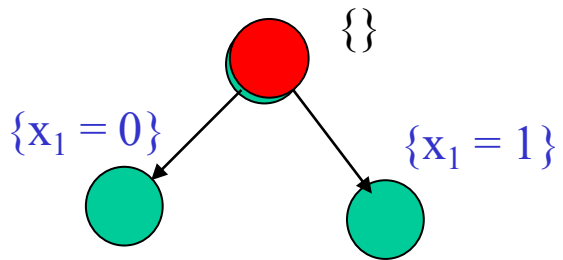
Queue:

Incumbent: none

Best Z^* : - inf

- Try to *fathom* (finish or prune):
 1. Infeasible?
 2. Worse than incumbent?
 3. Integer solution?

Example: B&B for BIPs



Solve:

$$\text{Max } Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

Subject to:

$$- 6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$$

$$- x_3 + x_4 \leq 1$$

$$- -x_1 + x_3 \leq 0$$

$$- -x_2 + x_4 \leq 0$$

$$- x_i \leq 1, x_i \geq 0, x_i \text{ integer}$$

$$Z = 16.5, x = \langle 0.8333, 1, 0, 1 \rangle$$

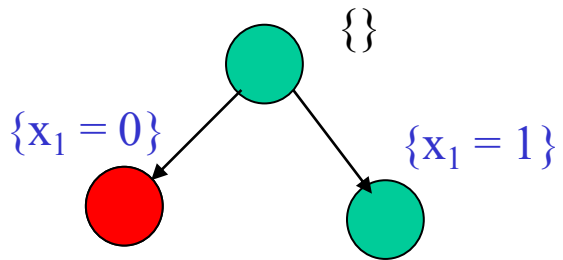
Queue: $\{x_1 = 0\} \{x_1 = 1\}$

Incumbent: none

Best Z^* : - inf

- Branch:
 1. Select unassigned x_i
 - Pick non-integer $\rightarrow x_1$
 2. Split on x_i

Example: B&B for BIPs



Solve:

$$\text{Max } Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

Subject to:

$$- 6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$$

$$- x_3 + x_4 \leq 1$$

$$- -x_1 + x_3 \leq 0$$

$$- -x_2 + x_4 \leq 0$$

$$- x_i \leq 1, x_i \geq 0, x_i \text{ integer}$$

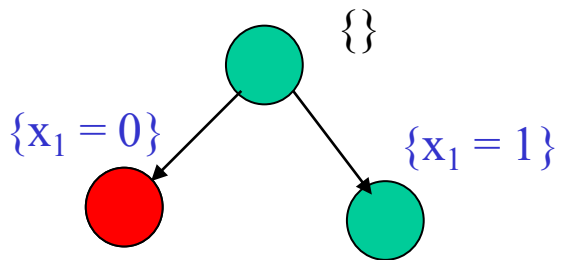
Queue: ~~{x₁ = 0}~~ {x₁ = 1}

Incumbent: none

Best Z*: - inf

• Dequeue $D_2 \equiv \{x_1 = 0\}$

Example: B&B for BIPs



Solve:

$$\text{Max } Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

Subject to:

$$- 6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$$

$$- x_3 + x_4 \leq 1$$

$$- x_1 + x_3 \leq 0$$

$$- x_2 + x_4 \leq 0$$

$$- x_i \leq 1, x_i \geq 0, x_i \text{ integer}$$

Queue: $\{x_1 = 1\}$

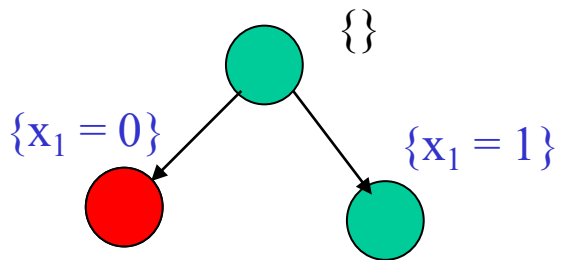
Incumbent: none

Best Z^* : - inf

• Bound $D_2 \equiv \{x_1 = 0\}$:

1. Constrain IP by $\{x_1 = 0\}$

Example: B&B for BIPs



Solve:

$$\text{Max } Z = 9 \mathbf{0} + 5x_2 + 6x_3 + 4x_4$$

Subject to:

$$- 6 \mathbf{0} + 3x_2 + 5x_3 + 2x_4 \leq 10$$

$$- x_3 + x_4 \leq 1$$

$$- \mathbf{0} + x_3 \leq 0$$

$$- x_2 + x_4 \leq 0$$

$$- x_i \leq 1, x_i \geq 0, x_i \text{ integer}$$

Queue: $\{x_1 = 1\}$

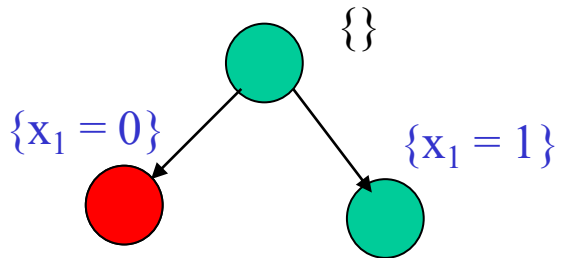
Incumbent: none

Best Z^* : - inf

• Bound $D_2 \equiv \{x_1 = 0\}$:

1. Constrain IP by $\{x_1 = 0\}$

Example: B&B for BIPs



Solve:

$$\text{Max } Z = 5x_2 + 6x_3 + 4x_4$$

Subject to:

$$3x_2 + 5x_3 + 2x_4 \leq 10$$

$$- x_3 + x_4 \leq 1$$

$$+ x_3 \leq 0$$

$$- x_2 + x_4 \leq 0$$

$$- x_i \leq 1, x_i \geq 0, x_i \text{ integer}$$

$$Z = 9, \quad x = \langle 0, 1, 0, 1 \rangle$$

Queue: $\{x_1 = 1\}$

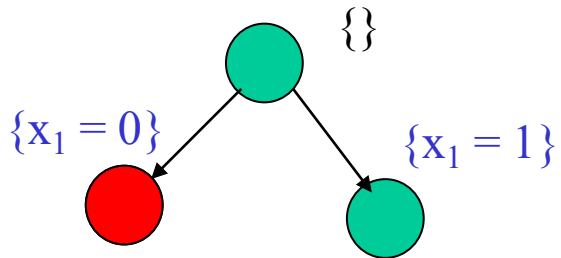
Incumbent: none

Best Z^* : - inf

• Bound $D_2 \equiv \{x_1 = 0\}$:

1. Constrain IP by $\{x_1 = 0\}$
2. Relax to LP
3. Solve LP

Example: B&B for BIPs



Solve:

$$\text{Max } Z = 5x_2 + 6x_3 + 4x_4$$

Subject to:

$$3x_2 + 5x_3 + 2x_4 \leq 10$$

$$- x_3 + x_4 \leq 1$$

$$+ x_3 \leq 0$$

$$- x_2 + x_4 \leq 0$$

$$- x_i \leq 1, x_i \geq 0, x_i \text{ integer}$$

$$Z = 9, \quad x = \langle 0, 1, 0, 1 \rangle$$

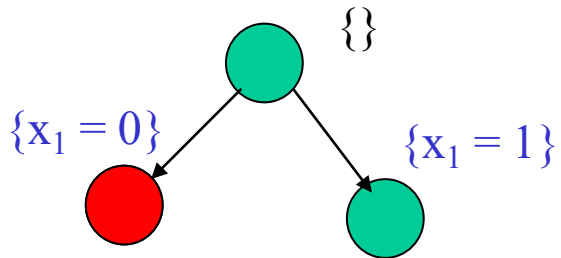
Queue: $\{x_1 = 1\}$

Incumbent: none

Best Z^* : - inf

- Try to fathom:
 1. Infeasible?
 2. Worse than incumbent?
 3. Integer solution?

Example: B&B for BIPs



Solve:

$$\text{Max } Z = 5x_2 + 6x_3 + 4x_4$$

Subject to:

$$3x_2 + 5x_3 + 2x_4 \leq 10$$

$$- x_3 + x_4 \leq 1$$

$$+ x_3 \leq 0$$

$$- x_2 + x_4 \leq 0$$

$$- x_i \leq 1, x_i \geq 0, x_i \text{ integer}$$

$$Z = 9, \quad \mathbf{x} = \langle 0, 1, 0, 1 \rangle$$

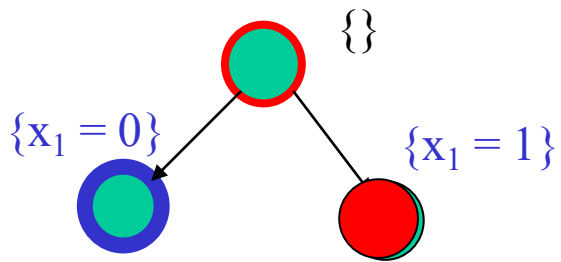
Queue: $\{x_1 = 1\}$

Incumbent: $\mathbf{x} = \langle 0, 1, 0, 1 \rangle$

Best Z^* : 9

- Try to fathom:
 1. Infeasible?
 2. Worse than incumbent?
 3. Integer solution?

Example: B&B for BIPs



Solve:

$$\text{Max } Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

Subject to:

$$- 6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$$

$$- x_3 + x_4 \leq 1$$

$$- x_1 + x_3 \leq 0$$

$$- x_2 + x_4 \leq 0$$

$$- x_i \leq 1, x_i \geq 0, x_i \text{ integer}$$

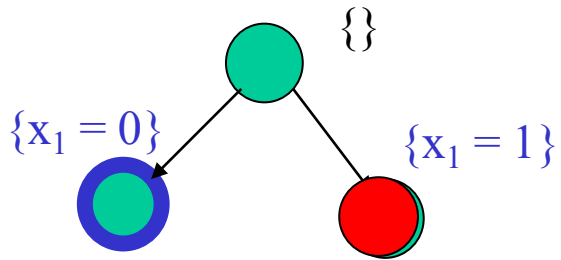
Queue: ~~$\{x_1 = 1\}$~~

Incumbent: $\mathbf{x} = \langle 0, 1, 0, 1 \rangle$

Best Z^* : 9

• Dequeue $D_3 \equiv \{x_1 = 1\}$

Example: B&B for BIPs



Solve:

$$\text{Max } Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

Subject to:

- $6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$
- $x_3 + x_4 \leq 1$
- $-x_1 + x_3 \leq 0$
- $-x_2 + x_4 \leq 0$
- $x_i \leq 1, x_i \geq 0, x_i$ **integer**

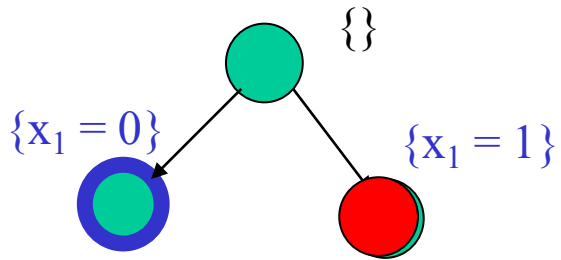
Queue:

Incumbent: $\mathbf{x} = \langle 0, 1, 0, 1 \rangle$

Best Z^* : **9**

• Bound $D_3 \equiv \{x_1 = 1\}$

Example: B&B for BIPs



Solve:

$$\text{Max } Z = \mathbf{9} + 5x_2 + 6x_3 + 4x_4$$

Subject to:

$$- \mathbf{6} + 3x_2 + 5x_3 + 2x_4 \leq 10$$

$$- x_3 + x_4 \leq 1$$

$$- \mathbf{-1} + x_3 \leq 0$$

$$- x_2 + x_4 \leq 0$$

$$- x_i \leq 1, x_i \geq 0, \text{ ~~x}_i \text{ integer}~~$$

$$Z = \mathbf{16.2}, \mathbf{x} = \langle \mathbf{1}, \mathbf{.8}, \mathbf{0}, \mathbf{.8} \rangle$$

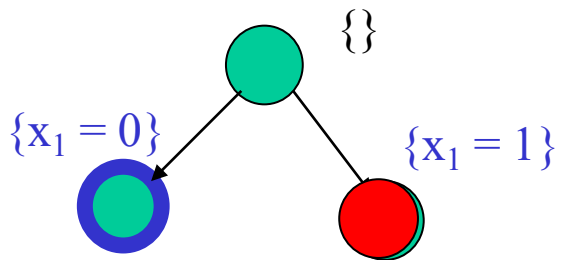
- Bound $D_3 \equiv \{x_1 = 1\}$

Queue:

Incumbent: $\mathbf{x} = \langle \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1} \rangle$

Best Z^* : $\mathbf{9}$

Example: B&B for BIPs



Solve:

$$\text{Max } Z = \mathbf{9} + 5x_2 + 6x_3 + 4x_4$$

Subject to:

$$- \mathbf{6} + 3x_2 + 5x_3 + 2x_4 \leq 10$$

$$- x_3 + x_4 \leq 1$$

$$- \mathbf{-1} + x_3 \leq 0$$

$$- x_2 + x_4 \leq 0$$

$$- x_i \leq 1, x_i \geq 0, \underline{x_i \text{ integer}}$$

$$Z = \mathbf{16.2}, \mathbf{x} = \langle \mathbf{1}, \mathbf{.8}, \mathbf{0}, \mathbf{.8} \rangle$$

Queue:

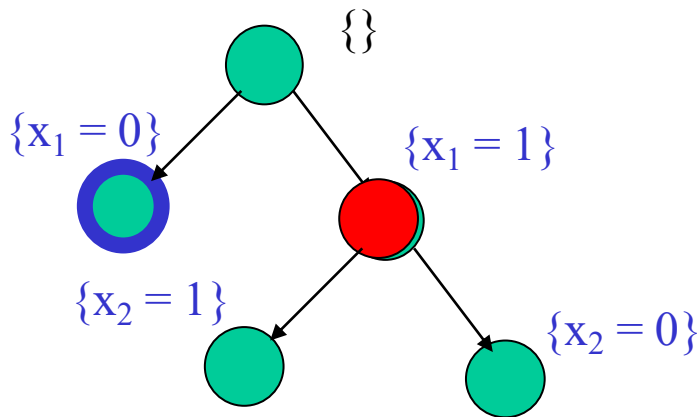
Incumbent: $\mathbf{x} = \langle \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1} \rangle$

Best Z^* : $\mathbf{9}$

• Try to fathom:

1. Infeasible?
2. Worse than incumbent?
3. Integer solution?

Example: B&B for BIPs



Solve:

$$\text{Max } Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

Subject to:

- $6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$
- $x_3 + x_4 \leq 1$
- $-x_1 + x_3 \leq 0$
- $-x_2 + x_4 \leq 0$
- $x_i \leq 1, x_i \geq 0, x_i$ **integer**

$$Z = 16.7, \mathbf{x} = \langle 1, .8, 0, .8 \rangle$$

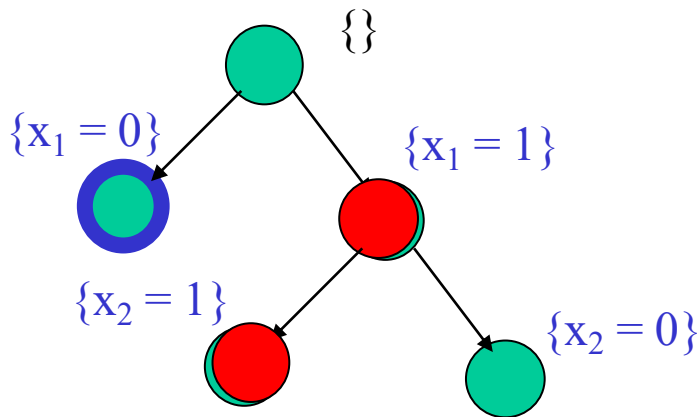
Queue: $\{x_1=1, x_2=1\} \{x_1=1, x_2=0\}$

• Branch on x_2

Incumbent: $\mathbf{x} = \langle 0, 1, 0, 1 \rangle$

Best Z^* : 9

Example: B&B for BIPs



Solve:

$$\text{Max } Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

Subject to:

- $6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$
- $x_3 + x_4 \leq 1$
- $-x_1 + x_3 \leq 0$
- $-x_2 + x_4 \leq 0$
- $x_i \leq 1, x_i \geq 0, x_i$ **integer**

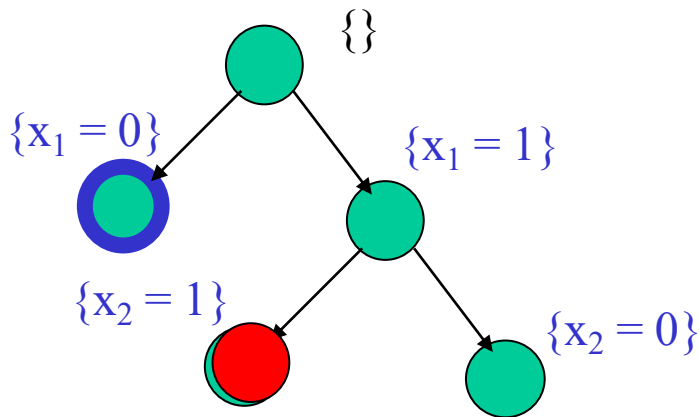
Queue: ~~$\{x_1=1, x_2=1\}$~~ $\{x_1=1, x_2=0\}$

Incumbent: $\mathbf{x} = \langle 0, 1, 0, 1 \rangle$

Best Z^* : **9**

- Branch on x_2
- Dequeue D_4

Example: B&B for BIPs



Solve:

$$\text{Max } Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

Subject to:

$$- 6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$$

$$- x_3 + x_4 \leq 1$$

$$- -x_1 + x_3 \leq 0$$

$$- -x_2 + x_4 \leq 0$$

$$- x_i \leq 1, x_i \geq 0, x_i \text{ integer}$$

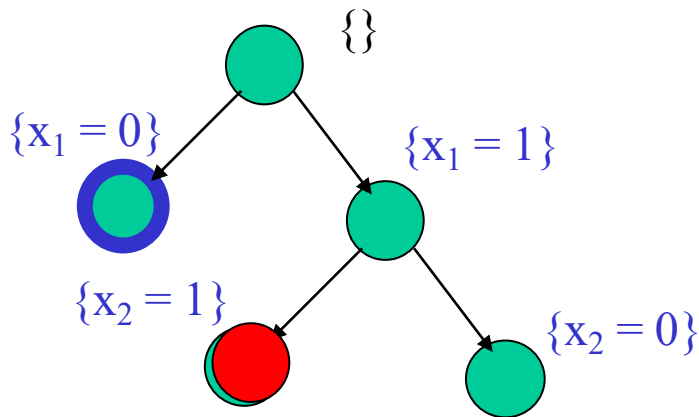
Queue: $\{x_1=1, x_2=0\}$

Incumbent: $\mathbf{x} = \langle 0, 1, 0, 1 \rangle$

Best Z^* : 9

• Bound $D_4 \equiv \{x_1 = 1, x_2 = 1\}$

Example: B&B for BIPs



Solve:

$$\text{Max } Z = 9 + 5 + 6x_3 + 4x_4$$

Subject to:

$$- 6 + 3 + 5x_3 + 2x_4 \leq 10$$

$$- x_3 + x_4 \leq 1$$

$$- -1 + x_3 \leq 0$$

$$- -1 + x_4 \leq 0$$

$$- x_i \leq 1, x_i \geq 0, x_i \text{ integer}$$

$$Z = 16, \mathbf{x} = \langle 1, 1, 0, .5 \rangle$$

Queue: $\{x_1=1, x_2=0\}$

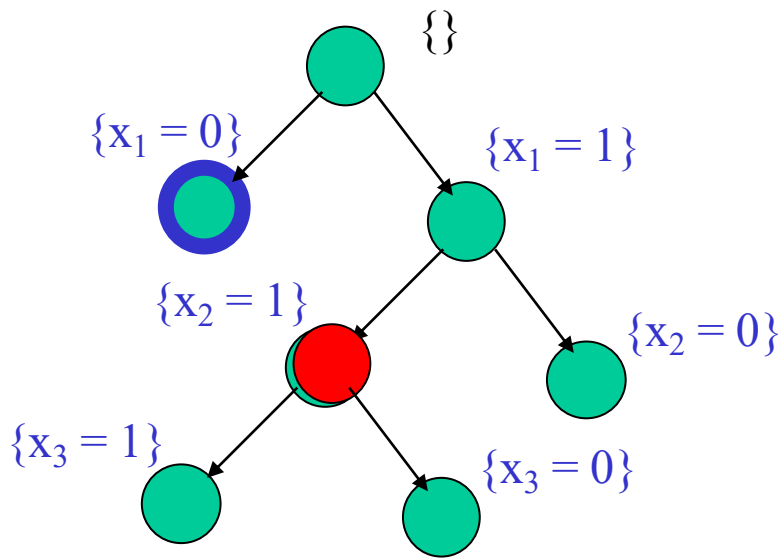
Incumbent: $\mathbf{x} = \langle 0, 1, 0, 1 \rangle$

Best Z^* : 9

• Try to fathom:

1. Infeasible?
2. Worse than incumbent?
3. Integer solution?

Example: B&B for BIPs



Solve:

$$\text{Max } Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

Subject to:

- $6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$
- $x_3 + x_4 \leq 1$
- $-x_1 + x_3 \leq 0$
- $-x_2 + x_4 \leq 0$
- $x_i \leq 1, x_i \geq 0, x_i$ ~~integer~~

$$Z = 16, \mathbf{x} = \langle 1, 1, 0, .5 \rangle$$

Queue: $\{\dots, x_2=0\} \{\dots, x_3=0\} \{\dots, x_2=0\}$

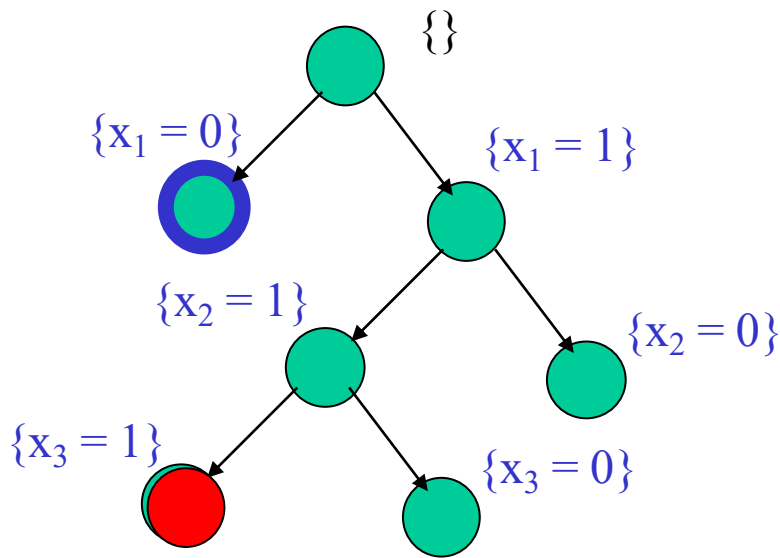
Incumbent: $\mathbf{x} = \langle 0, 1, 0, 1 \rangle$

Best Z^* : 9

• Branch

Correction: Should branch on x_4 , but this is more interesting.

Example: B&B for BIPs



... Fast Forward to D_5 ...

Queue: ~~{..., $x_3=1$ }~~ {..., $x_3=0$ } {..., $x_2=0$ }

• Dequeue D_5

Incumbent: $\mathbf{x} = \langle 0, 1, 0, 1 \rangle$

Best Z^* : 9

Solve:

$$\text{Max } Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

Subject to:

$$- 6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$$

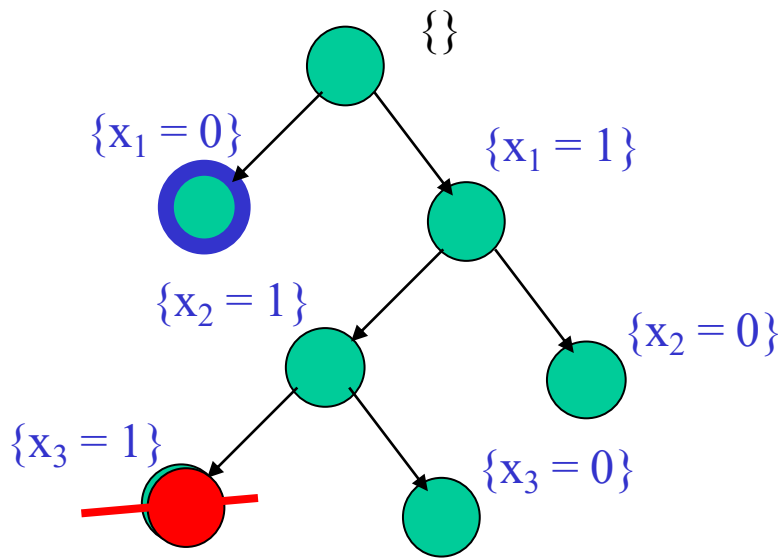
$$- x_3 + x_4 \leq 1$$

$$- x_1 + x_3 \leq 0$$

$$- x_2 + x_4 \leq 0$$

$$- x_i \leq 1, x_i \geq 0, x_i \text{ integer}$$

Example: B&B for BIPs



Solve:

$$\text{Max } Z = 9 + 5 + 6 + 4x_4$$

Subject to:

$$- 6 + 3 + 5 + 2x_4 \leq 10$$

$$- 1 + x_4 \leq 1$$

$$- -1 + 1 \leq 0$$

$$- -1 + x_4 \leq 0$$

$$- x_i \leq 1, x_i \geq 0, x_i \text{ integer}$$

Infeasible, No Solution

Queue: $\{\dots, x_3 = 0\} \{\dots, x_2 = 0\}$

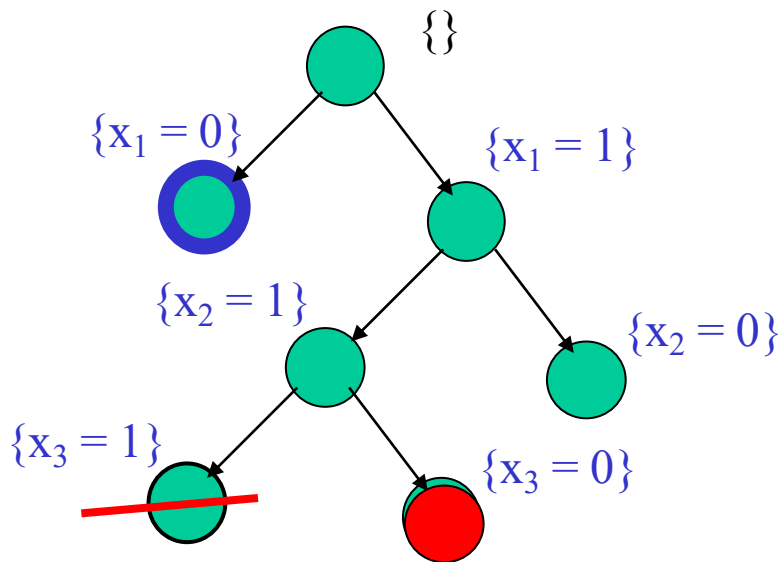
Incumbent: $\mathbf{x} = \langle 0, 1, 0, 1 \rangle$

Best Z^* : 9

• Try to fathom:

1. Infeasible?

Example: B&B for BIPs



Solve:

$$\text{Max } Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

Subject to:

$$- 6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$$

$$- x_3 + x_4 \leq 1$$

$$- x_1 + x_3 \leq 0$$

$$- x_2 + x_4 \leq 0$$

$$- x_i \leq 1, x_i \geq 0, x_i \text{ integer}$$

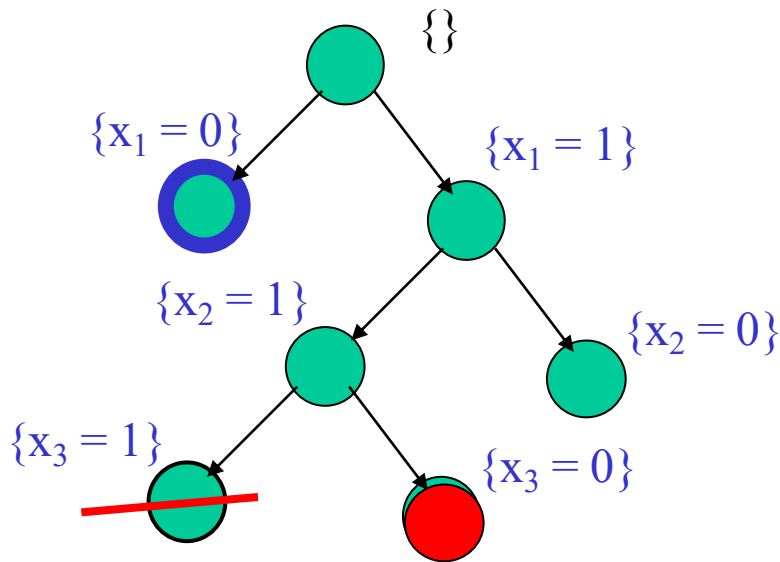
Queue: ~~{..., x₃ = 0}~~ {..., x₂ = 0}

Incumbent: $\mathbf{x} = \langle 0, 1, 0, 1 \rangle$

Best Z^* : 9

- Dequeue D_6
- Bound $D_6 \equiv \{x_1=1, x_2=1, x_3=0\}$

Example: B&B for BIPs



Solve:

$$\text{Max } Z = 9 + 5 + \quad + 4x_4$$

Subject to:

$$- 6 + 3 + \quad + 2x_4 \leq 10$$

$$+ x_4 \leq 1$$

$$- -1 \leq 0$$

$$- -1 + x_4 \leq 0$$

$$- x_i \leq 1, x_i \geq 0, x_i \text{ integer}$$

$$Z = 16, \mathbf{x} = \langle 1, 1, 0, .5 \rangle$$

Queue: $\{\dots, x_2 = 0\}$

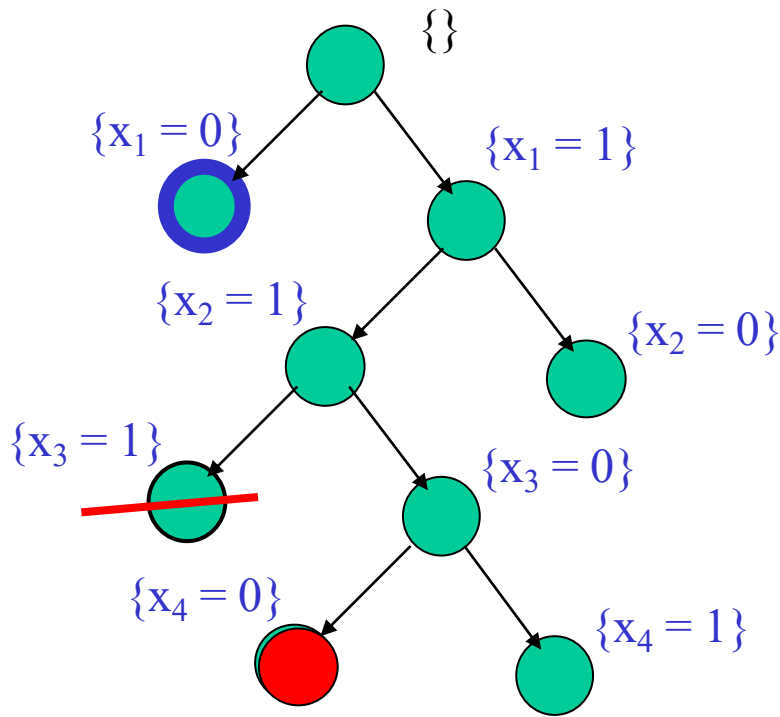
Incumbent: $\mathbf{x} = \langle 0, 1, 0, 1 \rangle$

Best Z^* : 9

• Try to fathom:

1. Infeasible?
2. Worse than incumbent?
3. Integer solution?

Example: B&B for BIPs



Solve:

$$\text{Max } Z = 9x_1 + 5x_2$$

Subject to:

$$-6x_1 + 3x_2 \leq 10$$

$$x_1 \leq 1$$

$$-x_1 \leq 0$$

$$-x_2 \leq 0$$

$$-x_i \leq 1, x_i \geq 0, x_i \text{ integer}$$

$$D_7: Z = 14, \mathbf{x} = \langle 1, 1, 0, 0 \rangle$$

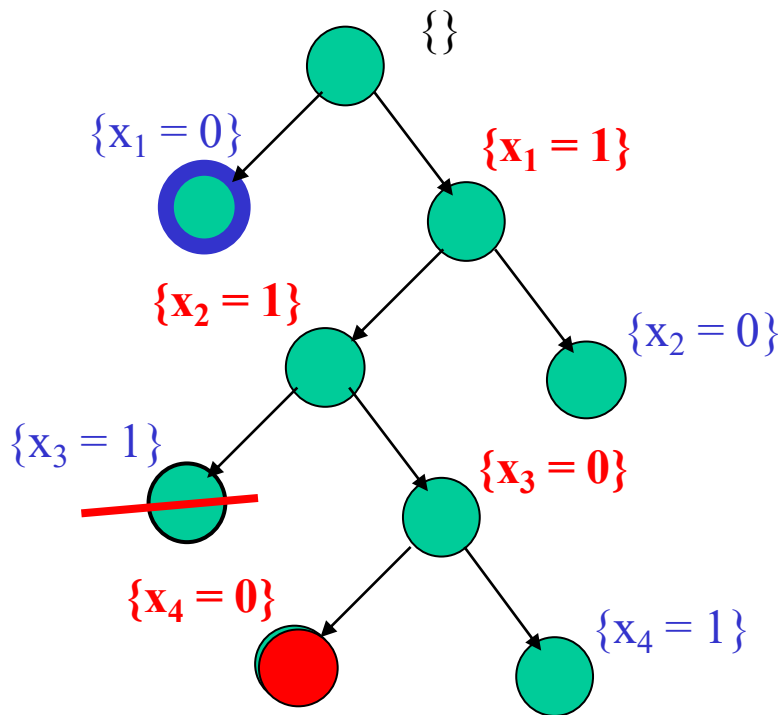
Queue: ~~$\{\dots, x_4=0\}$~~ $\{\dots, x_4=1\}$ $\{\dots, x_2=0\}$

Incumbent: $\mathbf{x} = \langle 0, 1, 0, 1 \rangle$

Best Z^* : 9

- Branch
- Dequeue D_7
- Bound D_7

Example: B&B for BIPs



... Next D_7 ...

Queue: $\{\dots, x_4=1\} \{\dots, x_2=0\}$

Incumbent: $\mathbf{x} = \langle 0, 1, 0, 1 \rangle$

Best Z^* : 9

Solve:

$$\text{Max } Z = 9x_1 + 5x_2$$

Subject to:

$$-6x_1 + 3x_2 \leq 10$$

$$x_1 \leq 1$$

$$-x_1 \leq 0$$

$$-x_2 \leq 0$$

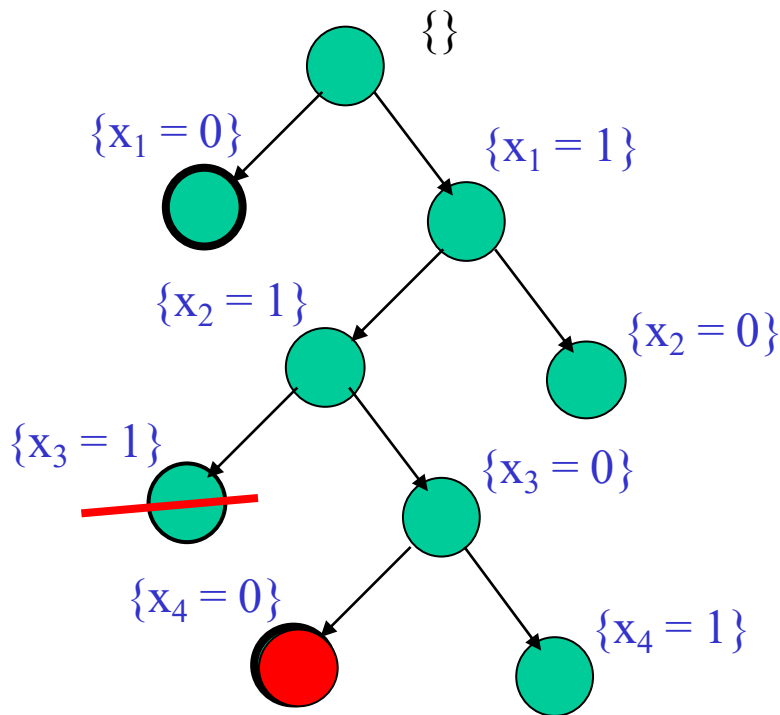
$$-x_i \leq 1, x_i \geq 0, x_i \text{ integer}$$

$D_7: Z = 14, \mathbf{x} = \langle 1, 1, 0, 0 \rangle$

• Try to fathom:

1. Infeasible?
2. Worse than incumbent?
3. Integer solution?

Example: B&B for BIPs



Solve:

$$\text{Max } Z = 9x_1 + 5x_2$$

Subject to:

$$-6x_1 + 3x_2 \leq 10$$

$$\leq 1$$

$$-x_1 \leq 0$$

$$-x_2 \leq 0$$

$$-x_i \leq 1, x_i \geq 0, x_i \text{ integer}$$

$$\mathbf{D_7: Z = 14, x = \langle 1, 1, 0, 0 \rangle}$$

Queue: $\{\dots, x_4=1\} \{\dots, x_2=0\}$

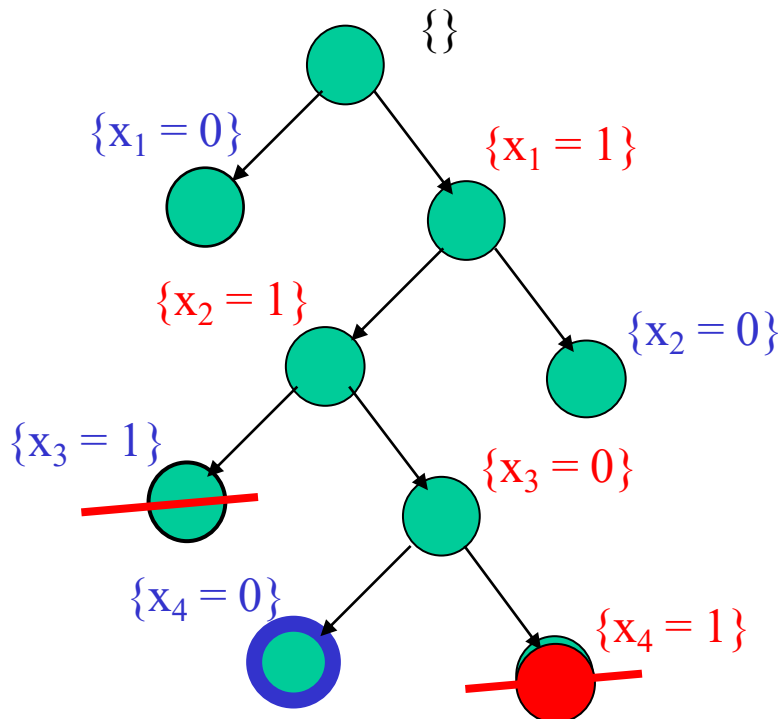
Incumbent: $\mathbf{x = \langle 1, 1, 0, 0 \rangle}$

Best Z^* : $\mathbf{14}$

• Try to fathom:

1. Infeasible?
2. Worse than incumbent?
3. Integer solution?

Example: B&B for BIPs



... Next D_8 ...

Queue: ~~{..., $x_4=1$ }~~ {..., $x_2=0$ }

Incumbent: $\mathbf{x} = \langle 1, 1, 0, 0 \rangle$

Best Z^* : 14

Solve:

$$\text{Max } Z = 9 \quad + 5 \quad + 4$$

Subject to:

$$- 6 \quad + 3 \quad + 2 \leq 10$$

$$+ 1 \leq 1$$

$$- -1 \leq 0$$

$$- -1 \quad + 1 \leq 0$$

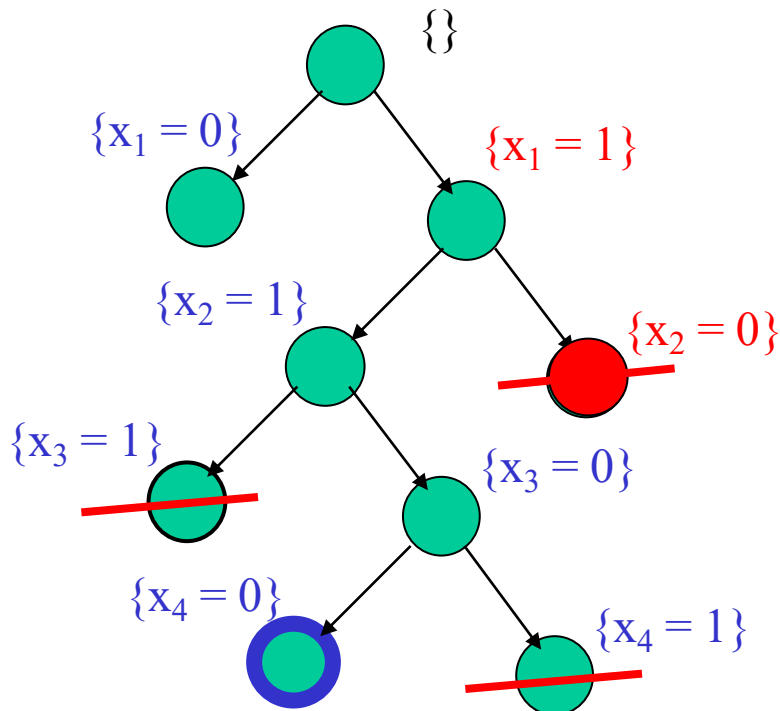
$$- x_i \leq 1, x_i \geq 0, x_i \text{ integer}$$

Infeasible, $\mathbf{x} = \langle 1, 1, 0, 1 \rangle$

• Try to fathom:

1. Infeasible?
2. Worse than incumbent?
3. Integer solution?

Example: B&B for BIPs



... Last D₉ ...

Queue: ~~{.., x₂=0}~~

Incumbent: $\mathbf{x} = \langle 1, 1, 0, 0 \rangle$

Best Z^* : 14

Solve:

$$\text{Max } Z = 9 \quad + 6x_3 + 4x_4$$

Subject to:

$$- 6 \quad + 5x_3 + 2x_4 \leq 10$$

$$- x_3 + x_4 \leq 1$$

$$- -1_1 + x_3 \leq 0$$

$$- -1 + x_4 \leq 0$$

$$- x_i \leq 1, x_i \geq 0, x_i \text{ integer}$$

Worse, $Z = 13.8$, $\mathbf{x} = \langle 1, 0, .8, 0 \rangle$

• Try to fathom:

1. Infeasible?
2. Worse than incumbent?
3. Integer solution?

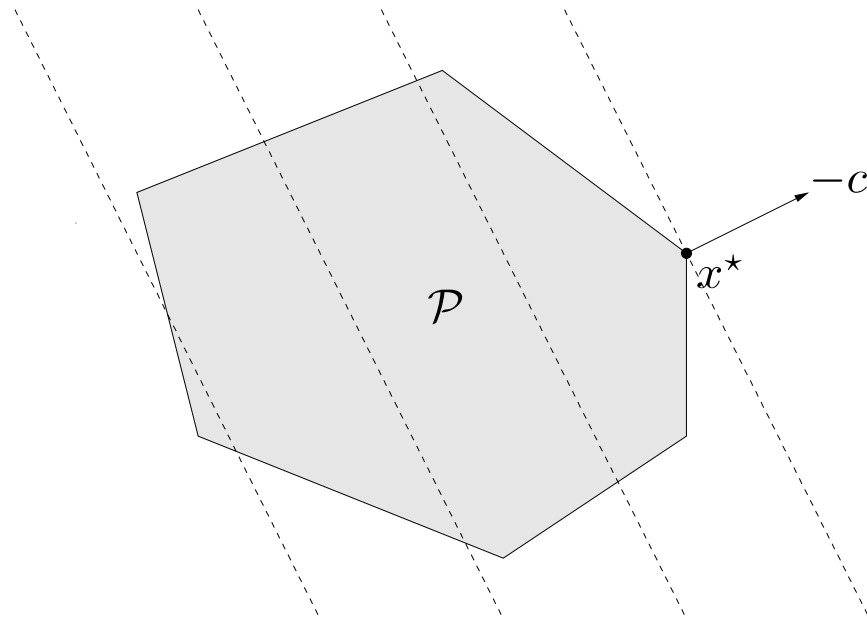
Solving nonlinear programs

Linear programs (LP)

$$\begin{array}{ll} \text{minimize} & c^T x + d \\ \text{subject to} & Gx \preceq h \\ & Ax = b, \end{array}$$

Affine objective (convex)

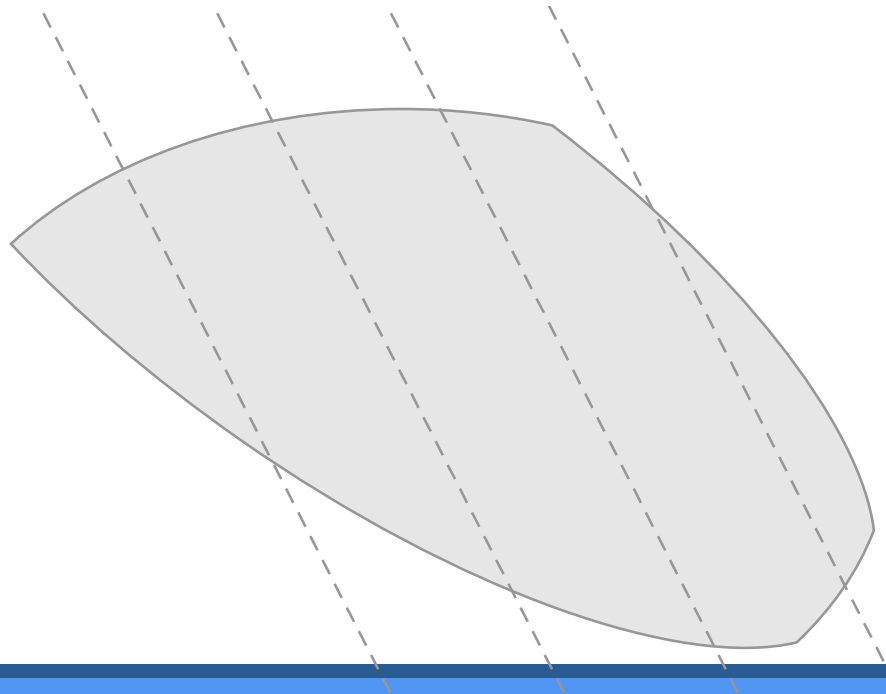
Intersection of half-spaces and hyperplanes



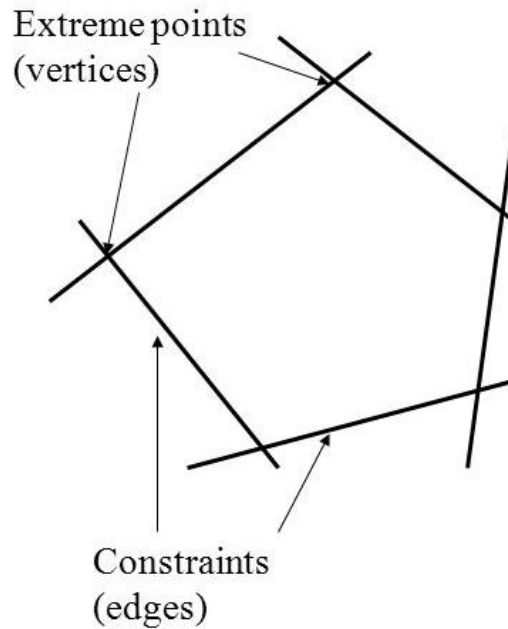
Nonlinear programs

$$\begin{array}{ll} \text{minimize} & c^T x + d \\ \text{subject to} & Gx \preceq h \\ & Ax = b, \end{array}$$

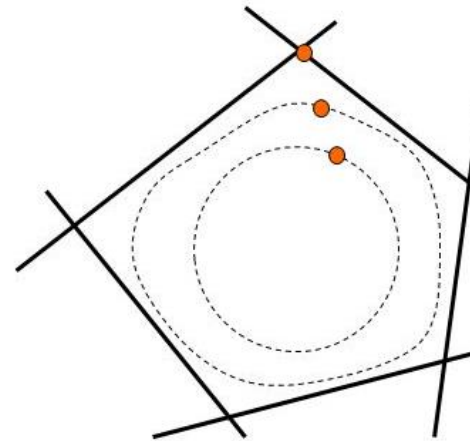
Solution not guaranteed to lie at a “vertex”, even with linear objectives.



Interior point methods



Simplex: search from vertex to vertex along the edges



Interior-point methods: go through the inside of the feasible space

Interior point methods

$$\min_{\mathbf{x}} f_0(\mathbf{x})$$

$$s.t. \quad f_i(\mathbf{x}) \leq 0, \quad i = 1, \dots, m$$

f_i convex, twice differentiable



$$\min_{\mathbf{x}} f_0(\mathbf{x}) + \sum_{i=1}^m I_-(f_i(\mathbf{x}))$$

$$I_-(x) = \begin{cases} 0 & x \leq 0 \\ \infty & x > 0 \end{cases}$$

Know how to do **unconstrained** optimization

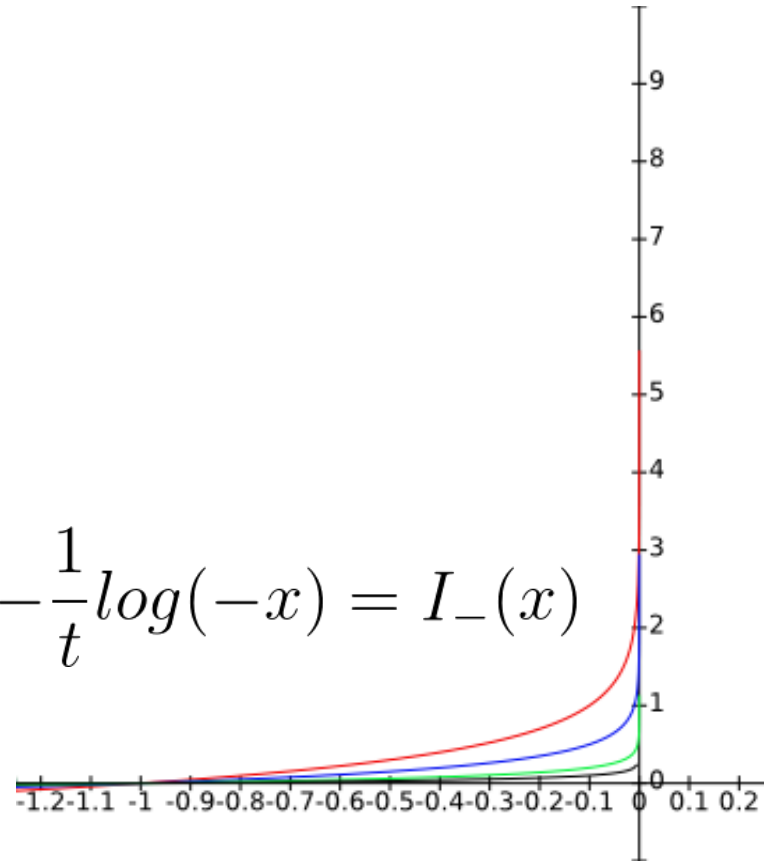
Idea: Penalize constraint violations!

Logarithmic marrier

$$\min_{\mathbf{x}} f_0(\mathbf{x}) + \sum_{i=1}^m I_-(f_i(\mathbf{x}))$$

$$I_-(x) = \begin{cases} 0 & x \leq 0 \\ \infty & x > 0 \end{cases}$$

$$\lim_{t \rightarrow \infty} -\frac{1}{t} \log(-x) = I_-(x)$$

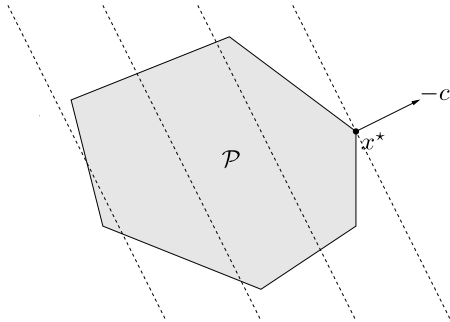


Original penalty: no gradients!

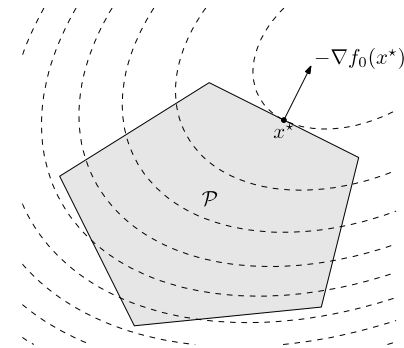
Approximate!

Some classes of convex optimization

Linear Programs (LP)



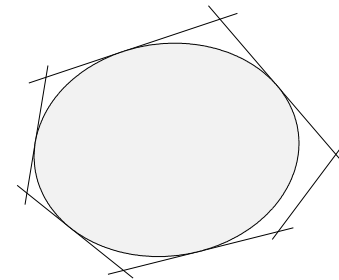
Quadratic Programs (QP)



Second Order Cone Programs (SOCP)

e.g. Robust LP (w/ uncertainty)

Semidefinite Programming (SDP)



Some classes of convex optimization problems

$LP \subseteq QP \subseteq SOCP \subseteq SDP \subseteq \text{Convex Problems}$

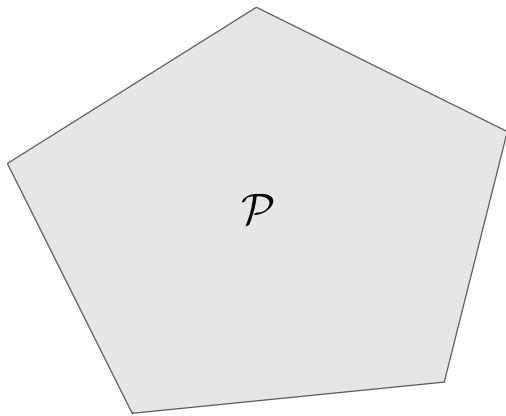
There are efficient specialized solvers for these problems

Quadratic optimization

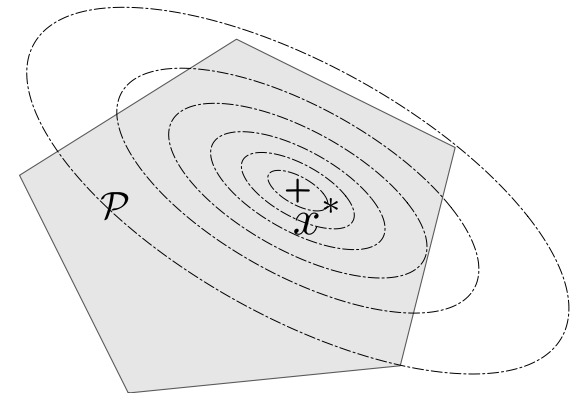
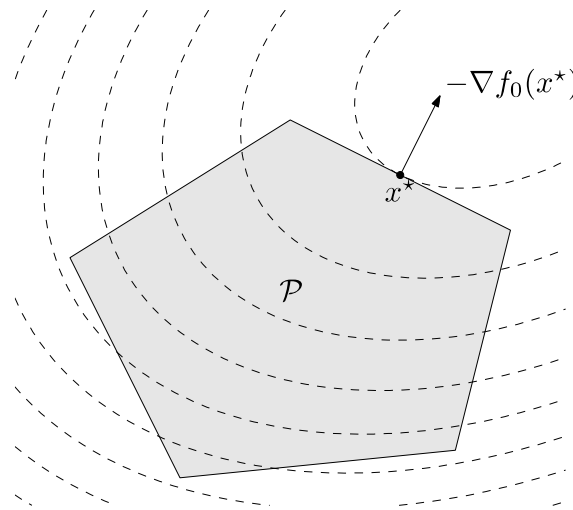
$$\begin{aligned} &\text{minimize} && (1/2)x^T P x + q^T x + r \\ &\text{subject to} && Gx \preceq h \\ &&& Ax = b, \end{aligned}$$

Objective function convex if P is PSD

Where is the optimal point?



feasible set



Next week

- Planning with uncertainty and continuous reward structure
 - Move from plans to policies
- Learning policies from experience